A KEY

TO

AN INTRODUCTORY COURSE OF

Plane Trigonometry and Logarithms.

(THIRD EDITION.)

BY

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PREFACE.

The present volume has been prepared in compliance with the expressed wishes of several mathematical teachers.

Judiciously employed, it is hoped that it will be found of service to private students also.

Of necessity, the solutions herein contained are not, in many cases, "set out" as the ordinary work of a student should be; but this can hardly be an inconvenience to those for whom the book is intended, especially when the number and variety of the examples worked out in the text-book are considered.

The author cannot sufficiently thank those gentlemen who have so generously and well assisted him in revising for the press—a labour, in such cases as the present, arduous in no ordinary degree. He trusts that any errors which may be found will be such as are evident at sight, and therefore cause no inconvenience.

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SOLUTIONS.

[Observe that, in these pages,—(1) In the case of identities, f represents the first side given, and l the other side. (2) In the case of equations, each solution begins with the first step after the statement of the equation.]

EXERCISE I.

1. $\frac{7}{21} = \frac{1}{3}$; $\frac{15}{85} = \frac{3}{17}$; $1 \div \frac{1}{2} = 1 \times 2 = 2$; $\frac{1}{2} \div \frac{3}{4} = \frac{1}{2} \times \frac{4}{3} = \frac{2}{3}$; $\frac{3}{4} \div 3\frac{1}{4} = \frac{3}{4} \times \frac{4}{13} = \frac{3}{13}$. **2.** 1 ft. : 1 yd. = 1 ft. : 3 ft. = 1 : 3; 1 yd. : $10\frac{1}{2}$ in. = 36 : $\frac{21}{2}$ = 72 : 21 = 24 : 7; 6 f. : $1\frac{1}{3}$ m. = $\frac{3}{4}$ m. : $\frac{4}{3}$ m. = 9 : 16. **3.** 1 ft. : 1 yd. = 1 ft. : 3 ft. = 1 : 3, 1 sq. ft. : 1 sq. yd. = 1 sq. ft. : 9 sq. ft. = 1 : 9, and $\frac{1}{3} = 3 \times \frac{1}{9}$.
4. 1 ft. : 1 in. = 12 in. : 1 in. = 12, 1 sq. ft. : 3×4 sq. in. = 144 sq. in. : 12 sq. in. = 12; ratio equal. **5.** $\dfrac{x \text{ hf.-crs.}}{x \text{ hf.-guins.}} = \dfrac{5x}{21x} = \dfrac{5}{21}$.
6. x : 5 cwt. = 3 : 7, $\therefore x = \dfrac{15 \text{ cwt}}{7} = 2$ cwt. 16 lbs.
7. Let x' = unit, then $270 : 18x = 3 : 2$, $\therefore x = 10'$.

EXERCISE II.

1. $\frac{1}{3}$ of $90° = 30°$; $\frac{3}{4}$ of $90° = 67°\,30'$; $\frac{17}{11}$ of $90° = 139°\,5'\,27''\cdot\dot{2}\dot{7}$.
2. $\frac{1}{2}$ of $100^g = 50^g$; $\frac{4}{3}$ of $100^g = 133^g\,33'\,33''\cdot\dot{3}$. $\frac{6}{7}$ of $100^g = 85^g\,77'\,42''\cdot86$.
3.

$$\begin{array}{r}
\cdot051544\dot{3} \text{ rt. } \angle \\
90 \\
\hline
4°\cdot638990 \\
60 \\
\hline
38'\cdot3394 \\
60 \\
\hline
20''\cdot364
\end{array}
\qquad\qquad
\begin{array}{r|l}
60 & \underline{\quad 15''} \\
60 & \underline{41\cdot25} \\
90 & \underline{41\cdot6875} \\
 & \cdot46319\dot{4} \text{ rt. } \angle
\end{array}$$

4. $81^\circ = \frac{10}{9} \times 81^g = 90^g$. $33^\circ\cdot75 = \frac{10}{9} \times 33^g\cdot75 = 37^g\,50'$.
$150^\circ = \frac{10}{9} \times 150^g = 166^g\,66'\,66''\cdot\dot{6}$. $74^\circ\,31'\,52''\cdot5 = 74^\circ\cdot53125 = \frac{10}{9} \times 74^g\cdot53125 = 82^g\,81'\,25''$. **5.** $150^g = \frac{9}{10} \times 150^\circ = 135^\circ$.
$18^g\,75' = \frac{9}{10} \times 18^\circ\cdot75 = 16^\circ\cdot875 = 16^\circ\,52'\,30''$. $63^g\,25'\,36''\cdot45 = \frac{9}{10} \times 63^\circ\cdot253645 = 56^\circ\cdot9282805 = 56^\circ\,55'\,41''\cdot8$.

6.
$$\begin{array}{r|l} 60 & 48''\cdot3 \\ 60 & 5'\cdot805 \\ 90 & 72^\circ\cdot09675 \\ & \cdot801075 \text{ rt. } \angle = 80^g\,10'\,75'' \end{array} \qquad \begin{array}{r|l} 60 & 1'' \\ 60 & 1\cdot0167 \end{array} \quad \&c.$$

7.

$\cdot153005$ rt. $\angle$	$\cdot0101011$ rt. $\angle$	$\cdot0404044$ rt. $\angle$
90	90	90
$13\cdot77045$ deg.	$0^\circ\cdot909099$	$3^\circ\cdot636396$
60	60	60
$46\cdot227$ min.	$54'\cdot54594$	$38'18376$
60	60	60
$13\cdot62$ sec.	$32''\cdot7564$	$11''0256$
$13^\circ\,46'\,13''\cdot62$		

CHAPTER II.

1. $\frac{1}{2}$ right angle $= \frac{1}{2} \times \frac{1}{2}\pi = \frac{1}{4} \times 3\cdot1416 = \cdot7854$. $\frac{2}{3}$ rt. $\angle = \frac{2}{3} \times \frac{1}{2}\pi = \frac{1}{3} \times 6\cdot2832 = 2\cdot0944$. $\frac{6}{7}$ rt. $\angle = \frac{6}{7} \times \frac{1}{2}\pi = \frac{1}{7} \times 9\cdot4248 = 1\cdot3464$.

2. $30^\circ = \frac{1}{3}$ of $90^\circ = \frac{1}{3} \times \frac{1}{2}\pi = \frac{1}{6}\pi = \cdot5236$. $22^\circ\,30' = \frac{1}{4} \times \frac{1}{2}\pi = \frac{1}{8}\pi = \cdot3927$.

$$90 : 156\tfrac{3}{4}^\circ = \frac{\pi}{2} : \frac{3\cdot1416 \times 627}{2 \times 4 \times 90} = 2\cdot7358.$$

$$90 : 96^\circ\,12'\,12'' = \frac{\pi}{2} : \frac{3\cdot1416 \times 28861}{2 \times 90 \times 300} = 1\cdot679069.$$

3. $$150^g = \frac{3}{2} \cdot \frac{\pi}{2} = \frac{3 \times 3\cdot1416}{4} = 2\cdot3562.$$

$$100^g : 73^g\,62'\,3''\cdot7 = \frac{\pi}{2} : \frac{3\cdot1416 \times 73\cdot62037}{2 \times 100} = 1\cdot1564288.$$

4. $\frac{5}{6}\pi = \frac{5}{3}$ of $90^\circ = 150^\circ$. $\frac{1}{2}\pi : \cdot5323 = 90^\circ : \dfrac{90^\circ \times \cdot5323}{1\cdot5708} =$ $30^\circ\,30'$ nearly. $\frac{1}{2}\pi : \cdot8447 = 90^\circ : \dfrac{90^\circ \times \cdot8447}{1\cdot5708} = 48^\circ\,23'\,51''\cdot5$.

5. $\frac{1}{5}\pi = \frac{1}{5} \times 200^g = 40^g$.
$$\tfrac{1}{2} : 1\cdot7482 = 100 : \frac{100^g \times 1\cdot7482}{1\cdot5708} = 111^g\,29'\,36''.$$

6. $150^\circ = \frac{10}{9} \times 150^g = \frac{1}{3} \times 500^g = 166^g\,66'\,66''\cdot7$.
$18^\circ\,7'\,14''\cdot16 = 18\cdot1206 = \frac{10}{9}$ of $18^g\cdot1206 = 20^g\cdot134 = 20^g\,13'\,40''$.

7. $125^g = \frac{9}{10}$ of $125° = \frac{1}{2} \times 225° = 112° 30'$.
$56^g 24' 25'' = 56^g{\cdot}2425 = \frac{1}{10} \times 9 \times 56{\cdot}2425 = 50°{\cdot}61825 = 50° 37' 5''{\cdot}7$.

8. Reqd. c. meas. $= \dfrac{4 \text{ ft. } 9 \text{ in.}}{7 \text{ ft. } 11 \text{ in.}} = \dfrac{57}{95} = \dfrac{3}{5} = {\cdot}6$.

9. $1{\cdot}7 = \dfrac{11 \text{ yds. } 1 \text{ ft.}}{\text{radius}}$, $\therefore$ rad. $= \dfrac{34 \text{ ft.}}{1{\cdot}7} = 20$ feet.

10. $\dfrac{\text{required arc}}{\text{radius}} =$ c. m. of $\angle = \frac{3}{4}$; $\therefore$ arc $= \frac{3}{4} \times 204$ m. $= 153$ m.

11. $\dfrac{\text{required arc}}{21} =$ c. meas. of $30° = \dfrac{\pi}{6} = \dfrac{22}{6{\cdot}7} = \dfrac{11}{21}$, $\therefore$ arc $= 11$.

12. Apply Euc. I. 32 and Cor. 1. The angle in case of—
(1) triangle $= \frac{1}{3}$ of $180° = 60°$ or $\frac{1}{3}\pi$.
(2) hexagon $= \frac{1}{6}(12 \times 90° - 4 \times 90°) = \frac{1}{6}$ of $8 \times 90° = 120°$ or $\frac{2}{3}\pi$.
(3) decagon $= \frac{1}{10}(20 \times 90° - 4 \times 90°) = \frac{1}{10}$ of $16 \times 90° = 144°$ or $\frac{4}{5}\pi$.
(4) quindecagon $= \frac{1}{15}(30 \times 90° - 4 \times 90°) = \frac{26}{15}$ of $90° = 156°$ or $\frac{13}{15}\pi$.

13. $\frac{1}{4}$ of $90 : 60 = 1 : 60 \times \frac{4}{90} = 2{\cdot}\dot{6}$. $\frac{1}{4}$ of $100 : 60 = 1 : 60 \times \frac{4}{100} = 2{\cdot}4$.

14. $\frac{1}{4}\pi : {\cdot}5236 = 1 : \dfrac{{\cdot}5236 \times 4}{3{\cdot}1416} = \dfrac{2}{3} = {\cdot}\dot{6}$.

EXERCISE III.

2. $c^2 = 9^2 + 12^2 = 81 + 144 = 225$, $\therefore c = 15$.
3. $a^2 = c^2 - b^2 = (25 + 20)(25 - 20) = 45 \times 5 = 225$, $\therefore a = 15$.
4. $c^2 = 100 + 576 = 676$, $\therefore c = 26$.
5. $c^2 = 49 + 81 = 130$, $\therefore c = \sqrt{130} =$ &c.
6. $a^2 = (10 + 9)(10 - 9) = 19$, $\therefore a = 4{\cdot}36\ldots$
7. $b^2 = (10 + 1)(10 - 1) = 99$, $\therefore b =$ &c.
8. sq. on diagonal $= (144 + 25)$ yds. $= 169$ yds., $\therefore$ diag. $= 13$ yds.
9. sq. on diagonal $= 2{\cdot}20^2$ sq. ft., $\therefore$ diag. $= 20\sqrt{2} =$ &c.
10. Let x ft. be side; then $2x^2 = (5\sqrt{2})^2 = 50$, $\therefore x = 5$.
11. sq. on breadth $= (30 + 27)(30 - 27)$ sq. yds. $= 57{\cdot}3$ sq. yds. $= 171$ sq. yds. $\therefore$ breadth $= \sqrt{171}$ yds. $= 13{\cdot}1\ldots$ yds.
12. Let x in. $=$ side, $\therefore x^2 (\frac{1}{2}x)^2 = 10^2$, $\therefore x^2 = \frac{1}{3} \times 20^2$,
$\therefore x = \frac{1}{3} \times 20\sqrt{3} =$ &c.
13. $BC^2 = (50 + 30)(50 - 30)$ sq. yds. $= 1600$ sq. yds., $\therefore BC = 40$.
14. sq. on dist. reqd. $= (125 + 75)(125 - 75)$ sq. yds. $= 10000$ sq. yds.
$\therefore$ dist. $= 100$ yds.
15. sq. on reqd. dist. $= (4 + \frac{64}{9})$ sq. m. $= \frac{100}{9}$ sq. m., $\therefore$ dist. $= 3\frac{1}{3}$ miles.
16. sq. on width $= (75 + 60)(75 - 60)$ sq. ft. $= 135{\cdot}15$ sq. ft. $= 9{\cdot}15^2$ sq. ft.
$\therefore$ width $= 3{\cdot}15$ ft. $= 45$ feet.

17. sq. on dist. $= [(3\cdot36)^2 + (1\cdot4)^2]$ sq. m. $= 13\cdot2496$ sq. m.,
$\therefore$ dist. $= 3\cdot64$ m.

18. Noting the tacks made by sailing vessel, square on distance in miles $= 30^2 + 40^2 = 2500$, $\therefore$ dist. $= 50$ miles.
$\therefore$ time of steamer $= \frac{50}{12}$ hrs. $= 4$ hrs. 10 min.

EXERCISE IV.

(3.) (i.) $l = c \times \dfrac{a}{c} = a.$ (ii.) $l = ac \times \dfrac{b}{c} \times \dfrac{b}{a} = b^2.$

(iii.) $l = ab^2 \times \dfrac{c}{b} \times \dfrac{c}{a} \times \dfrac{c}{b} = c^3.$

(iv.) $f = \left(\dfrac{a}{c} + \dfrac{b}{c}\right) \div \dfrac{b}{c} = \dfrac{a+b}{c} \times \dfrac{c}{b} = \dfrac{a+b}{b}.$

(v.) $f = \left(\dfrac{a}{c} + \dfrac{b}{c} + 1\right) \div \dfrac{a}{c} = \dfrac{a+b+c}{c} \times \dfrac{c}{a} = l.$

(vi.) $f = \dfrac{a^2}{b^2} - \dfrac{b^2}{a^2} = \dfrac{a^4 - b^4}{a^2b^2} = \dfrac{(a^2+b^2)(a^2-b^2)}{a^2b^2} = l.$

CHAPTER III.

1. $90° - 12° = 78°$; $90° - 25°\,4' = 64°\,56'$; $90° - 56°\,50'\,44'' = 33°\,9'\,16''$.
$90° - 88°\,58'\,58''\cdot89 = 1°\,1'\,1''\cdot11$.

2. $90° - (45° \mp A) = 90° - 45° \pm A = 45° \pm A$.

3. $\tan A = \tan(90° - A)$, $\therefore A = 90° - A$, $\therefore A = 45°$.

4. $\sin 5A = \sin(90° - A)$, $\therefore 6A = 90°$, $\therefore A = 15°$.

5. $\sec 3A = \sec(90° - 2A)$, $\therefore 5A = 90°$, $\therefore A = 18°$.

6. With the construction and diagram of Art. 21,

$\sec 30° = \dfrac{AB}{AD} = \dfrac{2a}{a\sqrt{3}} = \dfrac{2}{\sqrt{3}}$; $\operatorname{cosec} 30° = \dfrac{2a}{a} = 2$;

$\cot 30° = \dfrac{a\sqrt{3}}{a} = \sqrt{3}$; $\operatorname{vers} 30° = 1 - \cos 30° = 1 - \dfrac{a\sqrt{3}}{2a} = \frac{1}{2}(2 - \sqrt{3})$.

With construction of Art. 20,

$\sec 45° = \dfrac{AB}{AC} = \dfrac{a\sqrt{2}}{a} = \sqrt{2}$; $\operatorname{cosec} 45° = \dfrac{a\sqrt{2}}{a} = \sqrt{2}$;

$\cot 45° = \dfrac{a}{a} = 1$; $\operatorname{vers} 45° = 1 - \dfrac{a}{a\sqrt{2}} = \dfrac{\sqrt{2}-1}{\sqrt{2}}.$

With construction of Art. 21,

$\sec 60° = \dfrac{2a}{a} = 2$, $\operatorname{cosec} 60° = \dfrac{2a}{a\sqrt{3}} = \dfrac{2}{\sqrt{3}}$;

$\cot 60° = \dfrac{a\sqrt{3}}{a} = \sqrt{3}$, $\operatorname{vers} 60° = 1 - \dfrac{a}{2a} = \dfrac{1}{2}.$

7. $\sin^2 30° : \sin^2 45° : \sin^2 60° = \frac{1}{4} : \frac{1}{2} : \frac{3}{4} = 1 : 2 : 3.$

8. $f = \dfrac{1-\frac{1}{3}}{1+\frac{1}{3}} = \frac{1}{2} = l.$ **9.** $l = \dfrac{2 \times \frac{1}{\sqrt{3}}}{1-\frac{1}{3}} = \dfrac{2}{\sqrt{3}} \times \dfrac{3}{2} = \dfrac{1}{\sqrt{3}} = f.$

10. $f = 4 \times \frac{1}{8} - 3 \times \frac{1}{2} = \frac{1}{2} - \frac{3}{2} = -1.$

11. $f = \dfrac{3 \times \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}}}{1-\frac{3}{3}} = \dfrac{\sqrt{3} - \frac{1}{3\sqrt{3}}}{0} = l.$

12. $f = \dfrac{\frac{1}{\sqrt{2}} - \frac{1}{2}}{\frac{1}{\sqrt{2}} + \frac{1}{2}} = \dfrac{\sqrt{2}-1}{\sqrt{2}+1} = l.$ **13.** $f = \dfrac{\frac{\sqrt{3}}{2} + \frac{1}{2}}{\frac{\sqrt{3}}{2} - \frac{1}{2}} = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} = l.$

14. $f = 3 \times \frac{1}{3} - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2} = l.$

15. $l = \dfrac{\frac{1}{2}}{1} + \dfrac{1}{2} = 1,$ and $f = \dfrac{3}{2} - \dfrac{1}{2} = 1;$ $\therefore\ f = l.$

16. Make a rt. $\angle$d $\triangle$ ABC having a rt. $\angle$ C and $CA = 4$, $CB = 3$. Then $AB^2 = 25$; $\therefore AB = 5$; $\therefore \sin A = \dfrac{BC}{AB} = \frac{3}{5}$; $\cos A = \frac{4}{5}$ $\cot A = \frac{4}{3}$, $\sec A = \frac{5}{4}$, $\operatorname{cosec} A = \frac{5}{3}$, $\operatorname{vers} A = 1 - \frac{4}{5} = \frac{1}{5}$.

17. Make rt. $\angle$d $\triangle ABC$, with C right, $CB = 5$, $AB = 9$; then (Euc. I. 47) $AC = 2\sqrt{14}$, $\therefore$ $\tan A = \dfrac{5}{2\sqrt{14}}$; $\cos A = \dfrac{2\sqrt{14}}{9}$; $\operatorname{vers} A = 1 - \dfrac{2\sqrt{14}}{9} = \dfrac{1}{9}(9 - 2\sqrt{14}).$

18. Make rt. $\angle$d $\triangle$ with C rt., $CA = 1$, $AB = 2\sqrt{2}$; then $CB = \sqrt{7}$. $\therefore$ $\sin A = \dfrac{\sqrt{7}}{2\sqrt{2}} = \dfrac{1}{4}\sqrt{14}$, $\cot A = \dfrac{1}{\sqrt{7}}$.

19. Make rt. $\angle$d $\triangle$ with C rt., $AB = 1$, $CA = c$, $\therefore$ $CB = \sqrt{1-c^2}$. $\therefore$ $\sin A = \dfrac{CB}{AB} = \dfrac{\sqrt{1-c^2}}{1} = \sqrt{1-c^2}$; $\tan A = \dfrac{\sqrt{1-c^2}}{c}$, &c.

20. Let $\cot A = c = \dfrac{c}{1}$; and make rt. $\angle$ d $\triangle$ with C rt., $CA = c$,

$CB = 1$, $\therefore AB = \sqrt{1+c^2}$; $\therefore \sin A = \dfrac{1}{\sqrt{1+c^2}} = \dfrac{1}{\sqrt{(1+\cot^2 A)}}$,

$\cos A = \dfrac{c}{\sqrt{(1+c^2)}} =$ &c.

21. Sec $A = \frac{4}{9}$. Make ABC with C rt.; also $CA = 9$, $AB = 4$. This construction will fail.

22. Construct ABC with C rt.; and make $AB=p$, $CB=\sqrt{p^2-q^2}$,

$\therefore CA=q$, $\therefore \sin A=\dfrac{\sqrt{p^2-q^2}}{p}$, $\cos A=\dfrac{q}{p}$, $\tan A=\dfrac{\sqrt{p^2-q^2}}{q}$, &c.

CHAPTER IV.

1. $l=\cos A\dfrac{\sin A}{\cos A}=f.\qquad l=\sin A\dfrac{\cos A}{\sin A}=f.$

2. $f=\tan^2 A\dfrac{1}{\tan A}=l.\qquad l=\dfrac{1}{\cos A}\times\sin A=f.$

3. $l=\sqrt{\left(\dfrac{1}{\cos^2 A}\sin^2 A+1\right)}=\sqrt{(\tan^2 A+1)}=f.$

4. $l=\dfrac{1}{\sin A}\cos A=f.$ **5.** $f=\dfrac{\sin^4\theta}{\cos^2\theta}\times\dfrac{1}{\cos^2\theta}\times\dfrac{\cos^6\theta}{\sin^6\theta}=\dfrac{\cos^2\theta}{\sin^2\theta}$,

and $l=\dfrac{1}{\cos^2\theta}\times\dfrac{\cos^3\theta}{\sin^3\theta}\times\dfrac{\sin^2\theta}{1}\times\dfrac{\cos\theta}{\sin\theta}=\dfrac{\cos^2\theta}{\sin^2\theta}.$

6. $f=\cos(90°-A)\times\dfrac{1}{\sin(90°-A)}=l.$

7. $l=\sec A\times\dfrac{1}{\operatorname{cosec} A}=\dfrac{1}{\cos A}\times\dfrac{\sin A}{1}=f.$

8. $l=\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta=f.$

9. $f=\sin\theta\times\sin\theta+\cos\theta\times\cos\theta=l.$

10. $f=\dfrac{1}{\cos^2\theta}+\dfrac{1}{\sin^2\theta}=\dfrac{\sin^2\theta+\cos^2\theta}{\cos^2\theta\sin^2\theta}=\dfrac{1}{\cos^2\theta\sin^2\theta}=l.$

11. $f=\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}=\dfrac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}=\dfrac{1}{\cos\theta\sin\theta}=l.$

12. $f=(\cos^2\theta+\sin^2\theta)(\cos^2\theta-\sin^2\theta)=l.$

13. $f=\left(\cos\theta\times\dfrac{\sin\theta}{\cos\theta}\right)^2+\left(\sin\theta\times\dfrac{\cos\theta}{\sin\theta}\right)^2=\sin^2\theta+\cos^2\theta=1.$

14. $f=\left(\dfrac{\sin A}{\cos A}\times\cos A\right)^2+\left(\dfrac{\cos A}{\sin A}\times\sin A\right)^2=\sin^2 A+\cos^2 A=1.$

15. $f=\left(\dfrac{1}{\cos\theta}\dfrac{\cos\theta}{\sin\theta}\right)^2+\left(\dfrac{1}{\sin\theta}\dfrac{\sin\theta}{\cos\theta}\right)^2=\dfrac{1}{\sin^2\theta}+\dfrac{1}{\cos^2\theta}=\dfrac{\sin^2\theta+\cos^2\theta}{\sin^2\theta\cos^2\theta}=l.$

16. $f=\sin\theta\left(1-\dfrac{1}{\sin\theta}\right)=\sin\theta-1=l.$

17. $f=\sin A+\cos A\dfrac{\cos A}{\sin A}+\sin A\dfrac{\sin A}{\cos A}+\cos A$

$=\dfrac{\sin^2 A+\cos^2 A}{\sin A}+\dfrac{\sin^2 A+\cos^2 A}{\cos A}=l.$

18. $f = \dfrac{\operatorname{cosec}^2 A + \sec^2 A + 2\operatorname{cosec} A \sec A}{\operatorname{cosec}^2 A + \sec^2 A} \times \dfrac{\sin^2 A \cos^2 A}{\cos^2 A \cos^2 A}$

$= \dfrac{\cos^2 A + \sin^2 A + 2\sin A \cos A}{\sin^2 A + \cos^2 A} = l.$

19. $f = \dfrac{\sec A \cot A - \operatorname{cosec} A \tan A}{\cos A - \sin A} \times \dfrac{\sin A \cos A}{\sin A \cos A}$

$= \dfrac{\cos A - \sin A}{\cos A - \sin A} \times \dfrac{1}{\sin A \cos A} = l.$

20. $f = \sin^2 A\,(1 - \sin^2 B) - (1 - \sin^2 A)\sin^2 B = l.$

21. $f = \cos^2 A\,(1 - \sin^2 B) - (1 - \cos^2 A)\sin^2 B = l.$

22. $\cos A = \dfrac{\cos A}{\sin A}\sin A = \cot A\,\dfrac{1}{\operatorname{cosec} A} = \dfrac{\sqrt{(\operatorname{cosec}^2 A - 1)}}{\operatorname{cosec} A}.$

$\operatorname{cosec} A = \dfrac{1}{\sin A}\,\dfrac{\cos A}{\cos A} = \cot A \sec A = \dfrac{\sec A}{\tan A} = \dfrac{\sqrt{(\tan^2 A + 1)}}{\tan A}.$

23. $\tan A = \dfrac{\sqrt{(1 - \cos^2 A)}}{\cos A}$; $\quad \cot A = \dfrac{1}{\tan A} = \dfrac{1}{\sqrt{(\sec^2 A - 1)}}.$

24. $\sin\theta = \dfrac{\sin\theta}{\cos\theta}\cos\theta = \dfrac{\tan\theta}{\sec\theta} = \dfrac{\tan\theta}{\sqrt{(\tan^2\theta + 1)}}.$

$\operatorname{cosec}^2\alpha = \dfrac{1}{\sin^2\alpha} = \dfrac{1}{1 - \cos^2\alpha} = \dfrac{1}{(1 + \cos\alpha)(1 - \cos\alpha)} = \dfrac{1}{(2 - \operatorname{vers} A)\operatorname{vers} A}.$

25. $\sin A = \sqrt{(1 - \cos^2 A)} = \sqrt{(1 - \cdot 36)} = \sqrt{\cdot 64} = \cdot 8$;

$\tan A = \dfrac{\cdot 8}{\cdot 6} = 1\cdot\dot{3}$; $\quad \sec A = \dfrac{1}{\cdot 6} = 1\cdot\dot{6}$;

$\operatorname{cosec} A = \dfrac{1}{\cdot 8} = 1\cdot 25$; $\quad \cot A = \dfrac{\cdot 6}{\cdot 8} = \cdot 75$, &c.

26. $\cos A = \sqrt{(1 - \frac{121}{225})} = \sqrt{\frac{104}{225}} = \frac{2}{15}\sqrt{26}.\quad \tan A = \dfrac{11}{15} \times \dfrac{15}{2\sqrt{26}} = \dfrac{11}{52}\sqrt{26}.$

$\sec A = \dfrac{1}{\cos A} = \dfrac{15}{2\sqrt{26}} = \dfrac{15}{52}\sqrt{26}$, &c.

27. $\tan A = \dfrac{c}{a}$, $\quad \sec A = \sqrt{\left(\dfrac{c^2}{a^2} + 1\right)} = \dfrac{\sqrt{(c^2 + a^2)}}{a}.$

$\operatorname{cosec} A = \sqrt{\left(\dfrac{a^2}{c^2} + 1\right)} = \dfrac{\sqrt{(a^2 + c^2)}}{c}.$

$\sin A = \dfrac{c}{\sqrt{(a^2 + c^2)}}$, $\quad \cos A = \dfrac{a}{\sqrt{(c^2 + a^2)}}.$

28. $2\sin^2\theta = 1$, $\quad\therefore\ \sin\theta = \dfrac{1}{\sqrt{2}}$, $\quad\therefore\ \theta = \frac{1}{4}\pi.$

29. $2\tan^2\theta = \tan^2\theta + 1$, $\quad\therefore\ \tan^2\theta = 1$, $\quad\therefore\ \theta = \frac{1}{4}\pi.$

30. $2\cos^2\theta = 1\frac{1}{2}$, $\quad \cos\theta = \frac{1}{2}\sqrt{3}$, $\quad \therefore\ \theta = \frac{1}{6}\pi$.

31. $2\cos^2\theta - 3\cos\theta + 1 = 0$, $\quad \therefore\ \cos\theta = \frac{1}{4}(3 \pm \sqrt{1}) = 1$ or $\frac{1}{2}$, $\therefore\ \theta = \frac{1}{3}\pi$.

32. $1 + \cot^2\theta = 2\sec\theta\cot\theta = 2\operatorname{cosec}\theta$, $\quad \therefore\ \operatorname{cosec}\theta = 2$, $\quad \therefore\ \theta = \frac{1}{6}\pi$.

33. $2 - 2\cos^2\theta + 5\cos\theta = 4$, $\quad \therefore\ 2\cos^2\theta - 5\cos\theta + 2 = 0$,

$\therefore\ \cos\theta = \frac{1}{4}(5 \pm \sqrt{9}) = 2$ or $\frac{1}{2}$, $\quad \therefore\ \theta = \frac{1}{3}\pi$.

34. $\tan^2\theta + \tan\theta - 2 = 0$, $\therefore\ \tan\theta = \frac{1}{2}(-1 \pm \sqrt{9}) = 1$ or -2, $\therefore\ \theta = \frac{1}{4}\pi$.

35. $4\sin^4\theta - 7\sin^2\theta + 3 = 0$, $\quad \therefore\ \sin^2\theta = \frac{1}{8}(7 \pm \sqrt{1}) = 1$ or $\frac{3}{4}$,

$\therefore\ \sin\theta = \frac{1}{2}\sqrt{3}$, $\quad \therefore\ \theta = \frac{1}{3}\pi$.

36. $4\cos^2\theta - 3\cos\theta + \frac{1}{2} = 0$, $\therefore\ \cos\theta = \frac{1}{8}(3 \pm \sqrt{1}) = \frac{1}{2}$ or $\frac{1}{4}$, $\therefore\ \theta = \frac{1}{3}\pi$.

37. $2\operatorname{cosec}^2\theta - 5\operatorname{cosec}\theta + 2 = 0$, $\therefore\ \operatorname{cosec}\theta = \frac{1}{4}(5 \pm \sqrt{9}) = 2$ or $\frac{1}{2}$, $\therefore\ \theta = \frac{1}{6}\pi$.

38. $4\tan^2 A = \cot A \tan A = 1$, $\quad \therefore\ \tan A = \pm\frac{1}{2}$.

39. $25 - 25\cos^2 A = 16$, $\quad \therefore\ \cos^2 A = \frac{9}{25}$, $\quad \therefore\ \cos A = \pm\frac{3}{5}$.

40. $3\sin^2\alpha + 2\sin\alpha - 1 = 0$, $\quad \therefore\ \sin\alpha = \frac{1}{6}(-2 \pm \sqrt{16}) = \frac{1}{3}$ or -1.

41. $\tan^2 A - 4\tan A + 1 = 0$, $\quad \therefore\ \tan A = \frac{1}{2}(4 \pm \sqrt{12}) = 2 \pm \sqrt{3}$.

42. $m^2\sin^2 A = n^2(1 - \sin^2 A)$, $\quad \therefore\ \sin^2 A\,(m^2 + n^2) = n^2$,

$$\therefore\ \sin A = \pm\frac{n}{\sqrt{(m^2+n^2)}}.$$

43. $\dfrac{\sin\theta}{\cos^2\theta} = \dfrac{1}{\sqrt{2} - \sin\theta}$, $\quad \sin\theta\sqrt{2} - \sin^2\theta = \cos^2\theta = 1 - \sin^2\theta$.

$\therefore\ \sin\theta\sqrt{2} = 1$, $\ \therefore\ \sin\theta = \dfrac{1}{\sqrt{2}}$; $\quad \therefore\ \theta = \dfrac{\pi}{4}$.

CHAPTER V.

1. $\dfrac{a}{c} = \sin A$; $\qquad \therefore\ a = 5 \times \sin 30^\circ = 5 \times \frac{1}{2} = 2{\cdot}5$.

2. $\dfrac{a}{b} = \cot B$; $\qquad \therefore\ a = 10\sqrt{3}\cot 60^\circ = 10\sqrt{3} \times \dfrac{1}{\sqrt{3}} = 10$.

3. $\dfrac{c}{b} = \sec A$; $\qquad \therefore\ c = 20\sec 45^\circ = 20\sqrt{2}$.

4. $\cos B = \dfrac{a}{c} = \dfrac{3}{2\sqrt{3}} = \dfrac{\sqrt{3}}{2}$; $\qquad \therefore\ B = 30^\circ$.

5. $\tan A = \dfrac{a}{b} = \dfrac{6}{2\sqrt{3}} = \sqrt{3}$; $\qquad \therefore\ A = 60^\circ$.

6. $\sin B = \dfrac{b}{c} = \dfrac{5\sqrt{2}}{10} = \dfrac{1}{\sqrt{2}}$; $\qquad \therefore\ B = 45^\circ$.

7. $B = 90^\circ - A = 90^\circ - 30^\circ = 60^\circ$. $\quad \dfrac{a}{c} = \sin A$; $\ \therefore\ a = 50\sin 30^\circ =$

$50 \times \frac{1}{2} = 25$. $\quad \dfrac{b}{c} = \cos A$; $\ \therefore\ b = 50\cos 30^\circ = 50 \times \dfrac{\sqrt{3}}{2} = 25\sqrt{3}$

8. $A = 90° - B = 90° - 30° = 60°$. $\frac{b}{a} = \tan B$; $\therefore b = 1732 \times \frac{1}{\sqrt{3}} = \frac{1732}{1\cdot732} = 1000$. $\frac{c}{a} = \sec B$; $\therefore c = 1732 \times \frac{2}{\sqrt{3}} = \frac{1732 \times 2}{1\cdot732} = 2000$.

9. $\tan B = \frac{b}{a} = \frac{43\cdot3}{25} = 1\cdot732 = \sqrt{3}$; $\therefore B = 60°$.

$A = 90° - B = 30°$. $\frac{c}{a} = \sec B$; $\therefore c = 25 \times 2 = 50$.

10. $\sec A = \frac{c}{b} = \frac{50\cdot904}{36} = 1\cdot414 = \sqrt{2}$; $\therefore A = 45°$.

$B = 90° - A = 45°$. $\frac{a}{b} = \tan A$; $\therefore a = 36 \times 1 = 36$.

11. $\frac{a}{c} = \sin A$; $\therefore a = 100 \times \cdot145 = 14\cdot5$.

12. $\frac{a}{b} = \cot B$; $\therefore a = 3\cdot2 \times \cdot256 = \cdot8192$.

13. $\cos A = \frac{b}{c} = \frac{15\cdot5}{25} = \cdot62$; $\therefore A = 51° 41'$.

14. $\tan B = \frac{b}{a} = \frac{5529\frac{2}{3}}{8333\frac{1}{3}} = \frac{16589}{25000} = \cdot66356$; $\therefore B = 33° 34'$.

15. Taking fig. (6), $CA = CD \sec ACD = 433 \times 2 = 866$. $CB = CD \sec BCD = 433 \frac{2}{\sqrt{3}} = \frac{866}{1\cdot732} = 500$. $AB = AD + DB = CD \tan ACD + CD \tan BCD = 433\left(\sqrt{3} + \frac{1}{\sqrt{3}}\right) = 433 \frac{4}{\sqrt{3}} = \frac{1732}{1\cdot732} = 1000$.

16. Taking fig. (15), $PQH = 45° = PHQ$; $\therefore PQ = PH = x$ suppose; then $PR = x + 73\cdot2$; and $\because \frac{PR}{PH} = \cot 30°$, $\therefore \frac{x + 73\cdot2}{x} = \sqrt{3} = 1\cdot732$; $\therefore x(1\cdot732 - 1) = 73\cdot2$, or $\cdot732 \times x = 73\cdot2$; $\therefore x = 100$.

17. In fig. to Art. 30, (2), $CD = BC \cot\beta$; $\therefore AC = CD \tan\alpha = BC \tan\alpha \cot\beta = b\frac{\tan\alpha}{\tan\beta}$.

18. Let $CD = x$; then $BC = x\tan\beta$; $\therefore AC = x\tan\beta + a$; and $AC = x\tan\alpha$; $\therefore x\tan\alpha = x\tan\beta + a$; $\therefore x(\tan\alpha - \tan\beta) = a$; $\therefore x = \frac{a}{\tan\alpha - \tan\beta}$.

19. $DBC = 90° - BDC = 45° = BDC$; $\therefore CD = BD\cos 45° = \frac{70\cdot7}{1\cdot414} = 50$; $\therefore \tan ADC = \frac{CA}{CD} = \frac{86\cdot6}{50} = 1\cdot732 = \sqrt{3}$; $\therefore ADC = 60°$; $\therefore BDA = 60° - 45° = 15°$.

20. In ABC, fig. (14), each angle $= 60°$; and $\because BC = BD$, $BCD = D$. $\therefore 2D = BCD + D = ABC = 60°$; $\therefore ACD = 60° + 30° = 90°$; $\therefore CD = AC \tan A = a \tan 60° = a\sqrt{3}$.

CHAPTER VI.

*** The references in brackets—thus, (1), (2), &c.—are to the figures of Plate I.

1. In (1), let A be foot of person, CB the pillar. We have *given* $BC = 45$ ft., $A = 60°$; *required* AC. $AC = CB \cot A = 45 \text{ ft.} \times \cot 60° = 45 \text{ ft.} \times \frac{1}{\sqrt{3}} = \frac{15 \times 3}{\sqrt{3}} \text{ ft.} = 15\sqrt{3} \text{ ft.}$

2. In (2), let BC be tower, and CA the horizontal base. Given $AC = 100$ yds., $A = 30°$; required BC. $BC = AC \tan A = 100 \text{ yds.} \times \tan 30° = \frac{100}{\sqrt{3}} \text{ yds.} = \frac{100\sqrt{3}}{3} \text{ yds.}$

3. In (3), let S, S' be positions of ship, and L that of lighthouse, and let Se be eastward, $S'e$ southward. Given $SS' = 12$ m., $S'Se = 45°$, $eSL = 45°$, $eS'L = 15°$; required SL and $S'L$. Evidently $S'SL = 45° + 45° = 90°$, and the triangle is right-angled; also $SS'L = 45° + 15° = 60°$; $\therefore SL = SS' \tan SS'L = 12 \text{ m.} \times \tan 60° = 12 \text{ m.} \times \sqrt{3} = 12\sqrt{3} \text{ m.}$; and $S'L = SS' \sec SS'L = 12 \text{ m.} \times 2 = 24 \text{ m.}$

4. In (4), let CB be tower, BD the steeple, and A the point of observation. Given $DAC = 60°$, $BAC = 45°$; reqd. $DC : BC$. $BC = AC \tan 45° = AC$; $DC = AC \tan 60° = AC \times \sqrt{3}$; $\therefore DC : BC = AC\sqrt{3} : AC = \sqrt{3} : 1$.

5. In (5), let BD be pole, BC height of mound, A point of observation. Given $BAC = 30°$, $DAC = 60°$; reqd. to show $BD = 2BC$. $DC = AC \tan DAC = AC \tan 60° = AC\sqrt{3}$, $BC = AC \tan 30° = \frac{AC}{\sqrt{3}}$;

$$\therefore BD = DC - BC = AC\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) = AC\frac{3-1}{\sqrt{3}} = \frac{2AC}{\sqrt{3}} = 2BC.$$

6. In (6), let AB be canal, C the crossing of the roads, CD the path. Given $A = 30°$, $B = 60°$, $AC = 3$ m., CD perp. to AB; reqd. BC, CD. $A + B = 90°$, $\therefore ACB = 90°$, $\therefore BC = CA \tan A = 3 \text{ m.} \times \frac{1}{\sqrt{3}} = \sqrt{3} \text{ m.}$, and $CD = CA \sin A = 3 \text{ m.} \times \frac{1}{2} = 1\frac{1}{2} \text{ m.}$

7. In (7), let HC, CK be two positions of ladder, AB width of street, H, K the windows. Given $HC = CK = 30$ ft., $HA = 24$ ft., $HCK = 90°$; reqd. AB, BK. $AC^2 = HC^2 - HA^2 = 900 - 576 = 324$, $\therefore AC = 18$. Also $\because HCK = 90°$, $\therefore HCA + KCB = 90°$, $\therefore KCB = 90° - HCA = H$; $\therefore BC = CK \cos BCK = CH \cos H = HA = 24$ ft.; and $AB = 18 \text{ ft.} + 24 \text{ ft.} = 42$ ft.; also $BK = CK \sin BCK = CH \sin H = AC = 18$ ft.

8. In (8), let BC be steeple, A, D the two points of observation, DE parallel to AC. Given $AD = 30$ ft., $BAC = 45°$, $BDE = 30°$; reqd. BC, CA. $\because$ $BAC = 45°$, $\therefore$ $AC = CB = x$ ft. suppose; then $BE = BC - AD = x - 30$; $\therefore$ $\frac{x-30}{x} = \tan BDE = \tan 30° = \frac{1}{\sqrt{3}}$;
$\therefore$ $x(\sqrt{3}-1) = 30\sqrt{3}$; $\therefore$ $x = \frac{30\sqrt{3}}{\sqrt{3}-1} = \frac{30\sqrt{3}(\sqrt{3}+1)}{3-1} = 15(3+\sqrt{3})$; $\therefore$ $AC = BC = 15(3+\sqrt{3})$ ft.

9. In (3), let $S'S$, LS be courses of the two ships, Se eastward, $S'e$ southward. Given $S'Se = 45°$, $eSL = 45°$, $eS'L = 15°$; reqd. $LS : S'S$. Evidently $S'SL = 45° + 45° = 90°$, and the triangle is right-angled. Also $SS'L = 45° + 15° = 60°$. $\therefore$ $\frac{S'S}{LS} = \cot SS'L = \frac{1}{\sqrt{3}}$, or $S'S : LS = 1 : \sqrt{3}$.

10. In (10), let BC be mast, A point observed, BD horizontal. Given $BC = 125$ ft., $DBA = 6° 50' 34''$; reqd. AC. BAC = alternate $\angle$ DBA; $\therefore$ $AC = CB \cot BAC = \frac{CB}{\tan BAC} = \frac{BC}{\tan DBA} = \frac{125}{\cdot 12}$ ft. $= 1041\frac{2}{3}$ ft.

11. In (2), let AB be string, BC height of kite. Given $AB = 200$ yds., $A = 36°$; reqd. BC. $BC = AB \sin A = AB \sin 36° = 200$ yds. $\times \cdot 5878 = 117 \cdot 56$ yds.

12. In (11), let BC be wall, CA ditch, and AD the distance backward. Given $BC = 20$ ft., $BAC = 60°$, $BDC = 30°$; reqd. CA, AD. $CA = CB \cot BAC = 20$ ft. $\times \frac{1}{\sqrt{3}} = 11 \cdot 547$ ft. nearly. $CD = CB \cot D = 20\sqrt{3}$ ft. $= 34 \cdot 64$ ft.; $\therefore$ $AD = 34 \cdot 64$ ft. $- 11 \cdot 547$ ft. $= 23 \cdot 093$ ft.

13. In (12), let B be steeple, C the tower, and AD the distance walked. Given $AD = 5$ m., $ADB = 45°$, $ADC = 15°$; reqd. BC. $AC = AD \tan ADC = AD \tan 15° = AD \cot 75° = 5$ m. $\times \cdot 26 = 1 \cdot 3$ m.; $AB = AD = 5$ m.; $\therefore$ $BC = 1 \cdot 3$ m. $+ 5$ m. $= 6 \cdot 3$ m.

14. In (9), let A, B be the milestones, CD the height of the hill. Given $AB = 1760$ yds., $EDB = 12° 13'$, $EDA = 2° 45'$; reqd. $CD = x$ yards suppose. $AB = AC - BC = x \cot A - x \cot CBD = x(\cot ADE - \cot BDE) = x\left(\frac{1}{\cdot 048} - \frac{1}{\cdot 217}\right) = x\frac{\cdot 217 - \cdot 048}{\cdot 048 \times \cdot 217} = \frac{\cdot 169}{\cdot 010416}x = \frac{169}{10 \cdot 416}x$; $\therefore$ $x = \frac{10 \cdot 416}{169} AB = \frac{10 \cdot 416}{169} \times 1760 = 108 \cdot 47$ nearly; $\therefore$ $CD = 108 \cdot 47$ yds.

15. $a = a$ and $a \cot A = b$; $\therefore$ adding, $a(1 + \cot A) = a + b$; $\therefore$ $a = \frac{a+b}{1+\cot A}$. $c \sin A = a$, $c \cos A = b$; $\therefore$ $c(\sin A + \cos A) = a + b$; $\therefore$ $c = \frac{a+b}{\sin A + \cos A}$.

16. $a = a$, $a\tan B = b$; $\therefore a(1-\tan B) = a-b$; $\therefore a = \dfrac{a-b}{1-\tan B}$.

$c\cos B = a$, $c\sin B = b$; $\therefore c(\cos B - \sin B) = a-b$; $\therefore c = \dfrac{a-b}{\cos B - \sin B}$.

17. $a=a$, $a\cot A = b$, $a\operatorname{cosec} A = c$; $\therefore a(1+\cot A + \operatorname{cosec} A) = a+b+c$;

$\therefore a = \dfrac{a+b+c}{1+\cot A+\operatorname{cosec} A}$. $c\sin A = a$, $c\cos A = b$, $c = c$;

$\therefore c(\sin A + \cos A + 1) = a+b+c$; $\therefore c = \dfrac{a+b+c}{\sin A + \cos A + 1}$.

18. $a\cot A = b$, $a\operatorname{cosec} A = c$; $\therefore a(\cot A - \operatorname{cosec} A) = b-c$; $\therefore a = \dfrac{b-c}{\cot A - \operatorname{cosec} A}$.

CHAPTER VII.

1. (1) $360°+30° = 390°$. (2) $2.360°+70° = 790°$. (3) $3.360°+125° = 1080°+125° = 1205°$. (4) $4.360°+300° = 1440°+300° = 1740°$.

2. (1) At OP_3 the angle $= -(360°-125°) = -235°$; at OP it $= -(360°-30°) = -330°$. (2) $-360°-60° = -420°$. (3) $-2.360°-290° = -720°-290° = -1010°$.

3. A line turning from OA positively would have described on coming to OB for the 1st time $90°$, the 2nd time $360°+90°$, the 3rd time $2.360°+90°$. The line turning negatively, the angles would be $-270°$, $-360°-270°$, and $-2.360°-270°$. Hence the required angles are $90°$, $450°$, $810°$, $-270°$, $-630°$, $-990°$.

4. $50°$ in 1st; $250° = 180°+70°$, and is $\therefore$ in 3rd; $-450° = -360°-90°$, and is $\therefore$ between the 4th and 3rd; $650° = 360°+270°+20°$, and is $\therefore$ in 4th; $-850° = -2.360°-90°-40°$, and is $\therefore$ in 3rd; $-1000° = -2.360°-270°-10°$, and is $\therefore$ in 1st; $\frac{5}{3}\pi = \frac{3}{2}\pi+\frac{1}{6}\pi$, and is $\therefore$ in 4th; $-\frac{4}{3}\pi = -\pi-\frac{1}{3}\pi$, and is $\therefore$ in 2nd.

5. Let $AOP_2 = A$; then $\sin A = \dfrac{P_2N_2}{OP_2} = \dfrac{4}{5}$; $\cos A = \dfrac{ON_2}{OP_2} = \dfrac{-3}{5} = -\dfrac{3}{5}$.

$\tan A = \dfrac{4}{-3} = -\dfrac{4}{3}$; $\cot A = \dfrac{-3}{4} = -\dfrac{3}{4}$; $\sec A = \dfrac{5}{-3} = -\dfrac{5}{3}$;

$\operatorname{cosec} A = \dfrac{5}{4}$; $\operatorname{vers} A = 1+\dfrac{3}{5} = \dfrac{8}{5}$.

6. $\sin A = \dfrac{-y}{r} = -\dfrac{y}{r}$; $\cos A = \dfrac{-x}{r} = -\dfrac{x}{r}$; $\tan A = \dfrac{-y}{-x} = \dfrac{y}{x}$;

$\cot A = \dfrac{-x}{-y} = \dfrac{x}{y}$; $\sec A = \dfrac{r}{-x} = -\dfrac{r}{x}$; $\operatorname{cosec} A = \dfrac{r}{-y} = -\dfrac{r}{y}$;

$\operatorname{vers} A = 1+\dfrac{x}{r} = \dfrac{r+x}{r}$.

7. Let A be any angle (fig. to Art. 38); then, whatever position PN may have, $\sin A = \frac{PN}{OP}$ and $\operatorname{cosec} A = \frac{OP}{PN}$, by definition of these functions; $\therefore \sin A = \frac{1}{\operatorname{cosec} A}$ always. For all positions of ON, by defs., $\cos A = \frac{ON}{OP}$ and $\sec A = \frac{OP}{ON}$; $\therefore$ always $\cos A = \frac{1}{\sec A}$.

8. In all positions of OP, PON is a right-angled triangle having OP the hypotenuse; $\therefore$ by Geometry, always $PN^2 + ON^2 = OP^2$; $\therefore$ $\left(\frac{PN}{OP}\right)^2 + \left(\frac{ON}{OP}\right)^2 = 1$, or $\sin^2 A + \cos^2 A = 1$. Also $\left(\frac{PN}{ON}\right)^2 + 1 = \left(\frac{OP}{ON}\right)^2$ or $\tan^2 A + 1 = \sec^2 A$.

CHAPTER VIII.

1. Construct as in Art. 45. Then $\cot A = \frac{x}{y}$; hence

When A is	0 or 360°	in 1st Quadrant	90°	in 2nd Quadrant	180°	in 3rd Quadrant	270°	in 4th Quadrant
x is	r	+ve and decreasing	0	−ve and increasing	$-r$	− ve and decreasing	0	+ve and increasing
y is	0	+ve and increasing	r	+ve and decreasing	0	−ve and increasing	$-r$	−ve and decreasing
$\therefore$ cot A is	∞	+ve and decreasing	0	−ve and increasing	∞	+ve and decreasing	0	−ve and increasing

2. Evidently, in fig. to Art. 45, when AOP is $< 45°$, we have, in the $\triangle NOP$, $\angle OPN > \angle PON$, and $\therefore$ side $ON > PN$, $\therefore \frac{ON}{OP} > \frac{PN}{OP}$, or $\cos A > \sin A$. When AOP is between 45° and 135°, we have, in the $\triangle NOP$, $\angle PON > \angle OPN$, and $\therefore$ side $PN >$ side ON, and $\therefore$ $\frac{PN}{OP} > \frac{ON}{OP}$, or $\sin A > \cos A$.

3. In same fig., between 135° and 225°, we have, in $\triangle PON$, the $\angle PON < 45°$, and $\therefore \angle OPN > 45°$; $\therefore \angle PON$ is $< \angle OPN$, and $ON > PN$, and $\therefore \frac{ON}{PN} > \frac{PN}{ON}$, or $\cot A > \tan A$.

4. In same fig., $\sin A = \frac{PN}{OP}$, and in the right-angled triangle PON, OP is always the greatest side, being opposite to the right angle, $\therefore \frac{PN}{OP}$ cannot be > 1. Again, $\sec A = \frac{OP}{ON}$, which is never less than 1 for the reason just given.

5. In same fig., $\tan A = \frac{PN}{ON}$; and when A is near 90°, PN is very great and ON very small; but the latter changes sign on A passing through 90°; hence $\tan A_1$ and $\tan A_2$ are both very great, but the former + ve and the latter − ve. $\sec A = \frac{OP}{ON}$, of which OP has a fixed magnitude and ON is as just described; $\therefore$ $\sec A_1$ and $\sec A_2$ are both very great but differ in sign.

6. Cot A is very great in magnitude when ON is very great and PN very small and when they differ in sign. This happens when A approaches 180° and 360° closely.

7. Cosec A is very large and − ve when PN is very small and − ve, and $\therefore$ when A has just passed through 180° and also when it is approaching closely to 360°.

8. Construct as in Art. 45. $\sec A = \frac{r}{x}$, in which x alone varies. x varies from $-r$ at OA' through 0 to $+r$ at OA; after which it decreases again through 0 to $-r$. Hence the range of $\frac{r}{x}$ is from -1 through ∞ to $+1$, and the opposite.

9. $\operatorname{Cosec} A = \frac{r}{y}$, in which y alone varies; y is greatest when it $= \pm r$, $\therefore$ $\operatorname{cosec} A$ is least when it $= \pm 1$.

10. In fig. to Art. 45, $\tan A = \frac{y}{x}$, in which y and x both vary, but oppositely, the one increasing as the other decreases. As y is finite when x is approaching 0, and made as small as we like, $\frac{y}{x}$ is then as large as we like. As x is finite when y is as small as we like, $\frac{y}{x}$ may be taken as small as we like.

11. $\operatorname{Vers} A = 1 - \cos A$; and assuming the variations of $\cos A$ as known, and that $\cos A$ is never > 1,

When A is	0 or 2π	in 1st Quadrant	$\frac{1}{2}\pi$	in 2nd Quadrant	π	in 3rd Quadrant	$\frac{3}{2}\pi$	in 4th Quadrant
$\cos A$ is	1	+ve and decreasing	0	−ve and increasing	−1	−ve and decreasing	0	+ve and increasing
$\therefore$ vers A is	0	+ve and increasing	1	+ve and increasing	2	+ve and decreasing	1	+ve and decreasing

12. $\operatorname{Sec}^2\theta$ must vary similarly to $\sec\theta$, except that it increases and decreases more rapidly from being a square and is never negative. Hence we may say, assuming the variation of $\sec\theta$ as known,

When A is	0 or 2π	in 1st Quadrant	$\frac{1}{2}\pi$	in 2nd Quadrant	π	in 3rd Quadrant	$\frac{3}{2}\pi$	in 4th Quadrant
$\sec^2\theta$ is	1	increasing	∞	decreasing	1	increasing	∞	decreasing

CHAPTER IX.

1. $\cos 160° = -\cos(180° - 160°) = -\cos 20°$. $\tan 260° = \tan(180° + 80°) = \tan 80°$. $\sec(-200°) = \sec 200° = \sec(180° + 20°) = -\sec 20°$. $\text{cosec}(-1000°) = \text{cosec}(3 \,.\, 360° - 1000°) = \text{cosec}\, 80°$.

2. $\sin(\frac{3}{2}\pi - \alpha) = \sin[\pi + (\frac{1}{2}\pi - \alpha)] = -\sin(\frac{1}{2}\pi - \alpha) = -\cos\alpha$; $\tan(\frac{3}{2}\pi - \alpha) = \tan(\frac{1}{2}\pi - \alpha) = \cot\alpha$; $\cot(\frac{3}{2}\pi + \theta) = -\cot[2\pi - (\frac{3}{2}\pi + \theta)] = -\cot(\frac{1}{2}\pi - \theta) = -\tan\theta$; $\sec(\frac{3}{2}\pi + \theta) = \sec(\frac{1}{2}\pi - \theta) = \text{cosec}\,\theta$.

3. $\cos 135° = -\cos 45° = -\dfrac{1}{\sqrt{2}}$; $\cot 225° = \cot 45° = 1$;

$\text{cosec}(-510°) = -\text{cosec}\, 510° = -\text{cosec}\, 150° = -\text{cosec}\, 30° = -2$.

4. $\sin(2n\pi + \frac{2}{3}\pi) = \sin\frac{2}{3}\pi = \sin\frac{1}{3}\pi = \frac{1}{2}\sqrt{3}$; $\cos(-2n\pi + \frac{1}{6}\pi) = \cos\frac{1}{6}\pi = \frac{1}{2}\sqrt{3}$; $\tan(2n\pi + \pi - \frac{1}{4}\pi) = \tan(\pi - \frac{1}{4}\pi) = -\tan\frac{1}{4}\pi = -1$;

$\cot[2n\pi + (\frac{3}{2}\pi - \frac{5}{6}\pi)] = \cot\frac{2}{3}\pi = -\cot\frac{1}{3}\pi = -\dfrac{1}{\sqrt{3}}$.

5. $-\frac{1}{2}\sqrt{3} = -\cos 30° = \cos(180° - 30°)$, or $\cos(180° + 30°) = \cos 150°$ or $\cos 210°$, and $\cos 150° = \cos(360° + 150°)$; $\therefore$ required angles are $150°$, $210°$, $510°$.

6. $\cot^2\theta = 1$, $\therefore$ $\cot\theta = \pm 1$; and $1 = \cot\frac{1}{4}\pi = \cot(\pi + \frac{1}{4}\pi)$; also $-1 = \cot(\pi - \frac{1}{4}\pi)$ or $\cot(2\pi - \frac{1}{4}\pi)$; $\therefore$ the four least positive angles are $\frac{1}{4}\pi$, $\frac{3}{4}\pi$, $\frac{5}{4}\pi$, $\frac{7}{4}\pi$; the lines which contain these also contain $-\frac{7}{4}\pi$, $-\frac{5}{4}\pi$, $-\frac{3}{4}\pi$, $-\frac{1}{4}\pi$, which are the least negative angles.

7. $2\sin^2 A + \sin A - 1 = 0$; $\therefore \sin A = \frac{1}{4}[-1 \pm \sqrt{(1+8)}] = \frac{1}{4}(-1 \pm 3) = \frac{1}{2}$ or -1, and $\frac{1}{2} = \sin 30° = \sin 150°$, $-1 = \sin 270°$; $\therefore$ required angles are $30°$, $150°$, $270°$.

8. $2(1 - \cos^2 A) = 3\cos A$, $\therefore$ $2\cos^2 A + 3\cos A - 2 = 0$, $\therefore \cos A = \frac{1}{4}[-3 \pm \sqrt{(9+16)}] = \frac{1}{4}(-3 \pm 5) = \frac{1}{2}$ or -2; and $\frac{1}{2} = \cos 60° = \cos 300°$, -2 is not a cosine; $\therefore$ the angles are $60°$, $300°$.

9. $\tan^2\theta + \dfrac{1}{\tan^2\theta} = \dfrac{10}{3}$, $\therefore$ $3\tan^4\theta - 10\tan^2\theta + 3 = 0$, $\therefore$ $\tan^2\theta = \frac{1}{6}[10 \pm \sqrt{(100 - 36)}] = \frac{1}{6}(10 \pm 8) = 3$ or $\frac{1}{3}$, $\therefore \tan\theta = \pm\sqrt{3}$ or $\pm\dfrac{1}{\sqrt{3}}$; $\therefore$ angles are $60°$, $120°$, $30°$, $150°$.

10. Let A be the angle. Then $\sin(180° - A) = \sin(90° - A)$, $\therefore$ $\sin A = \cos A$, $\therefore$ $A = 45°$.

11. $\cos\left(\dfrac{0 \,.\, \pi}{2} - \theta\right) = \cos(-\theta) = \cos\theta$; $\cos\left(\dfrac{1 \,.\, \pi}{2} - \theta\right) = \sin\theta$.

$\cos\left(\dfrac{2 \,.\, \pi}{2} - \theta\right) = \cos(\pi - \theta) = -\cos\theta$; $\cos\left(\dfrac{3 \,.\, \pi}{2} - \theta\right) = -\cos(\frac{1}{2}\pi - \theta) = -\sin\theta$, $\cos\left(\dfrac{4 \,.\, \pi}{2} - \theta\right) = \cos(2\pi - \theta) = \cos(-\theta)$,

which is the first case over again. The rest come over again on taking higher values of n.

12. $\sin A = \cos(90° - A) = \cos[-(90° - A)] = \cos(A - 90°)$, $\cos A = \sin(90° - A) = -\sin[-(90° - A)] = -\sin(A - 90°)$.

EXERCISE V.

2. $l = \cos 90^\circ \cos A + \sin 90^\circ \sin A = f$; $l = \sin 90^\circ \cos A \pm \cos 90^\circ \sin A = f$; $l = -(\cos 180^\circ \cos A \pm \sin 180^\circ \sin A) = f$;
$l = -(\sin 2\pi \cos\theta - \cos 2\pi \sin\theta) = f$.

$$\cos\tfrac{1}{2}\pi = \cos\tfrac{1}{4}\pi\cos\tfrac{1}{4}\pi - \sin\tfrac{1}{4}\pi\sin\tfrac{1}{4}\pi = \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2 = 0.$$

$$\cos 0 = \cos\tfrac{1}{4}\pi\cos\tfrac{1}{4}\pi + \sin\tfrac{1}{4}\pi\sin\tfrac{1}{4}\pi = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1.$$

3. $f = \sin(45^\circ - 30^\circ) = \dfrac{1}{\sqrt{2}}\cdot\dfrac{\sqrt{3}}{2} - \dfrac{1}{\sqrt{2}}\cdot\dfrac{1}{2} = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$; and
$\sin 15^\circ = \cos(90^\circ - 15^\circ) = \cos 75^\circ$.
$f = \cos(45^\circ - 30^\circ) = \dfrac{1}{\sqrt{2}}\cdot\dfrac{\sqrt{3}}{2} + \dfrac{1}{\sqrt{2}}\cdot\dfrac{1}{2} = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$; and
$\cos 15^\circ = \sin(90^\circ - 15^\circ) = \sin 75^\circ$.

EXERCISE VI.

Answers to **2** and **3** are written out at sight.

5. Let $\alpha = 22\tfrac{1}{2}^\circ$, then $2\cos^2\alpha = 1 + \cos 2\alpha = 1 + \dfrac{1}{\sqrt{2}} = 1 + \dfrac{\sqrt{2}}{2} = \dfrac{2+\sqrt{2}}{2}$, $\therefore \cos\alpha = \tfrac{1}{2}\sqrt{(2+\sqrt{2})}$.

6. Let $\alpha = 11\tfrac{1}{4}^\circ$, then $2\sin^2\alpha = 1 - \cos 2\alpha = 1 - \tfrac{1}{2}\sqrt{(2+\sqrt{2})} = \tfrac{1}{2}[2 - \sqrt{(2+\sqrt{2})}]$, $\therefore \sin\alpha = \tfrac{1}{2}\sqrt{[2 - \sqrt{(2+\sqrt{2})}]}$.

7. $\cot(A+B) = \dfrac{\cos A\cos B - \sin A\sin B}{\sin A\cos B + \cos A\sin B} = \dfrac{\cot A\cot B - 1}{\cot A + \cot B}$, on dividing numerator and denominator by $\sin A\sin B$.

$\cot(A-B) = \dfrac{\cos A\cos B + \sin A\sin B}{\sin A\cos B - \cos A\sin B} = \dfrac{\cot A\cot B + 1}{\cot B - \cot A}$, similarly.

$\cot 2A = \cot(A+A) = \dfrac{\cot A\cot A - 1}{\cot A + \cot A} = \dfrac{\cot^2 A - 1}{2\cot A}$.

8. $\tan 75^\circ = \tan(45^\circ + 30^\circ) = \dfrac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ\tan 30^\circ} \times \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} = \dfrac{(\sqrt{3}+1)^2}{3-1} = 2 + \sqrt{3}$.

CHAPTER X.

1. $f = \sin 45^\circ\cos A + \cos 45^\circ\sin A = \dfrac{1}{\sqrt{2}}\cos A + \dfrac{1}{\sqrt{2}}\sin A = l$.

2. $f = \cos 30^\circ\cos A + \sin 30^\circ\sin A = \tfrac{1}{2}\sqrt{3}\cos A + \tfrac{1}{2}\sin A = l$.

3. $f = 1 - \cos(A+30^\circ) - [1 - \cos(A-30^\circ)] = \cos(A-30^\circ) - \cos(A+30^\circ) = \cos A\cos 30^\circ + \sin A\sin 30^\circ - \cos A\cos 30^\circ + \sin A\sin 30^\circ = 2\times\tfrac{1}{2}\sin A = l$.

4. $f = \dfrac{2\sin^2 A}{2\sin A\cos A} = \dfrac{\sin A}{\cos A} = l.$ **5.** $f = \dfrac{2\sin^2 A}{2\cos^2 A} = l.$

6. $f = \dfrac{\tan 45° + \tan A}{1 - \tan 45° \tan A} = l.$ **7.** $f = \dfrac{\tan A - \tan 45°}{1 + \tan A \tan 45°} = l.$

8. $f = \dfrac{\sin A}{\cos A} + \dfrac{\cos A}{\sin A} = \dfrac{\sin^2 A + \cos^2 A}{\cos A \sin A} = \dfrac{2}{2\sin A\cos A} = \dfrac{2}{\sin 2A} = l.$

9. $f = \dfrac{\sin A}{\cos A} - \dfrac{\cos A}{\sin A} = \dfrac{\sin^2 A - \cos^2 A}{\cos A \sin A} = \dfrac{-2(\cos^2 A - \sin^2 A)}{2\sin A\cos A}$

$= \dfrac{-2\cos 2A}{\sin 2A} = l.$

10. $f = \dfrac{\sin A}{\cos A} - \dfrac{\sin B}{\cos B} = \dfrac{\sin A\cos B - \cos A\sin B}{\cos A\cos B} = l.$

11. $f = \dfrac{\cos A}{\sin A} - \dfrac{\sin B}{\cos B} = \dfrac{\cos A\cos B - \sin A\sin B}{\sin A\cos B} = l.$

12. $l = \dfrac{\dfrac{2\sin A}{\cos A}}{1 + \dfrac{\sin^2 A}{\cos^2 A}} = \dfrac{\dfrac{2\sin A}{\cos A}}{\dfrac{\cos^2 A + \sin^2 A}{\cos^2 A}} = \dfrac{2\sin A\cos A}{1} = f.$

13. $l = \dfrac{1 - \dfrac{\sin^2 A}{\cos^2 A}}{1 + \dfrac{\sin^2 A}{\cos^2 A}} = \dfrac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = f.$

14. $l = 1 - \dfrac{2}{\operatorname{cosec}^2 A} = 1 - 2\sin^2 A = f.$

15. $l = \dfrac{\dfrac{\cos\frac{1}{2}A}{\sin\frac{1}{2}A} - \dfrac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A}}{\dfrac{\cos\frac{1}{2}A}{\sin\frac{1}{2}A} + \dfrac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A}} = \dfrac{\cos^2\frac{1}{2}A - \sin^2\frac{1}{2}A}{\cos^2\frac{1}{2}A + \sin^2\frac{1}{2}A} = f.$

16. $l = \cos 2(45° - \frac{1}{2}A) = \cos(90° - A) = f.$

17. $45° + A$ is complement of $45° - A$,

$\therefore\ l = 2\sin(45° - A)\cos(45° - A) = \sin(90° - 2A) = f.$

18. $f = 2\left(1 - \dfrac{\sin A\cos 2A}{\cos A\sin 2A}\right) = 2\,\dfrac{\cos A\sin 2A - \sin A\cos 2A}{\cos A\sin 2A}$

$= \dfrac{2\sin(2A - A)}{\cos A \,.\, 2\sin A\cos A} = \dfrac{1}{\cos^2 A} = l.$

19. $l = \dfrac{1}{1 + \dfrac{\sin 2A\sin 4A}{\cos 2A\cos 4A}} = \dfrac{\cos 2A\cos 4A}{\cos 2A\cos 4A + \sin 2A\sin 4A}$

$= \dfrac{\cos 2A\cos 4A}{\cos(4A - 2A)} = f.$

20. $f = (\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B) = \sin^2 A\cos^2 B$
$- \cos^2 A\sin^2 B = \sin^2 A(1 - \sin^2 B) - (1 - \sin^2 A)\sin^2 B = l.$

21. $f = \sin^2 A \cos^2 B - \cos^2 A \sin^2 B$
$= (1 - \cos^2 A) \cos^2 B - \cos^2 A (1 - \cos^2 B) = l.$

22. $f = \cos^2 A \cos^2 B - \sin^2 A \sin^2 B$
$= \cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B = l.$

23. $f = \cos^2 A \cos^2 B - \sin^2 A \sin^2 B$
$= (1 - \sin^2 A) \cos^2 B - \sin^2 A (1 - \cos^2 B) = l.$

24. $$f = \frac{\cos\theta \cos\frac{1}{4}\pi - \sin\theta \sin\frac{1}{4}\pi}{\cos\theta \cos\frac{1}{4}\pi + \sin\theta \sin\frac{1}{4}\pi} = \frac{\frac{1}{\sqrt{2}}(\cos\theta - \sin\theta)}{\frac{1}{\sqrt{2}}(\cos\theta + \sin\theta)} = \frac{(\cos\theta - \sin\theta)^2}{\cos^2\theta - \sin^2\theta}$$
$$= \frac{\cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta}{\cos 2\theta} = \frac{1 - \sin 2\theta}{\cos 2\theta} = l.$$

25. $$l = \left\{\frac{\tan\frac{1}{4}\pi + \tan\frac{1}{2}\alpha}{1 - \tan\frac{1}{4}\pi \tan\frac{1}{2}\alpha}\right\}^2 = \left(\frac{1 + \dfrac{\sin\frac{1}{2}\alpha}{\cos\frac{1}{2}\alpha}}{1 - \dfrac{\sin\frac{1}{2}\alpha}{\cos\frac{1}{2}\alpha}}\right)^2 = \left\{\frac{\cos\frac{1}{2}\alpha + \sin\frac{1}{2}\alpha}{\cos\frac{1}{2}\alpha - \sin\frac{1}{2}\alpha}\right\}^2$$
$$= \frac{\cos^2\frac{1}{2}\alpha + \sin^2\frac{1}{2}\alpha + 2\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha}{\cos^2\frac{1}{2}\alpha + \sin^2\frac{1}{2}\alpha - 2\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha} = f.$$

26. $$l = \sin\theta\,\frac{1}{\sec 2\phi} + \cos\theta\,\frac{1}{\operatorname{cosec} 2\phi} = \sin\theta\cos 2\phi + \cos\theta\sin 2\phi = f.$$

27. $$f = \frac{\sin(\alpha + 30^\circ)\sin(\alpha - 30^\circ)}{\cos(\alpha + 30^\circ)\cos(\alpha - 30^\circ)} = \frac{\sin^2\alpha - \sin^2 30^\circ}{\cos^2\alpha - \sin^2 30^\circ} = \frac{\sin^2\alpha - \frac{1}{4}}{\cos^2\alpha - \frac{1}{4}}$$
$$= \frac{4\sin^2\alpha - 2 + 1}{4\cos^2\alpha - 2 + 1} = \frac{1 - 2(1 - 2\sin^2\alpha)}{1 + 2(2\cos^2\alpha - 1)} = l.$$

28. $f = (\cos^2\alpha - \sin^2\alpha)(\cos^4\alpha + \cos^2\alpha\sin^2\alpha + \sin^4\alpha) =$
$$\cos 2\alpha\left[(\cos^2\alpha + \sin^2\alpha)^2 - \cos^2\alpha\sin^2\alpha\right] = \cos 2\alpha\left(1 - \frac{4\sin^2\alpha\cos^2\alpha}{4}\right) = l.$$

29. $$l = \sin(\alpha + \gamma)\cos 2(\beta + \gamma)\frac{\cos(\alpha + \gamma)}{\sin(\alpha + \gamma)} - \cos(\alpha + \gamma)\sin 2(\beta + \gamma) \times$$
$$\frac{\sin(\alpha + \gamma)}{\cos(\alpha + \gamma)} = \cos 2(\beta + \gamma)\cos(\alpha + \gamma) - \sin 2(\beta + \gamma)\sin(\alpha + \gamma)$$
$= \cos(2\beta + 2\gamma + \alpha + \gamma) = f.$

30. $$f = \frac{\sin(A + B)\cos C + \cos(A + B)\sin C}{\cos A\cos B\cos C} =$$
$$\frac{\sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C}{\cos A\cos B\cos C} = l.$$

31. $$f = \frac{\cos(A + B)\cos C - \sin(A + B)\sin C}{\cos A\cos B\cos C} =$$
$$\frac{\cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C}{\cos A\cos B\cos C} = l.$$

22. $f =$
$$\frac{\sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C}{\sin A\sin B\sin C} = l.$$

33. $f =$

$$\frac{\cos A\cos B\cos C-\sin A\sin B\cos C-\sin A\cos B\sin C-\cos A\sin B\sin C}{\sin A\sin B\sin C} = l.$$

34. $f =$

$$\frac{\sin A\cos B\cos C+\cos A\sin B\cos C+\cos A\cos B\sin C-\sin A\sin B\sin C}{\cos A\cos B\cos C-\sin A\sin B\cos C-\sin A\cos B\sin C-\cos A\sin B\sin C}$$

$$= \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan A\tan C-\tan B\tan C}$$

(by dividing numerator and denominator by $\cos A\cos B\cos C$).

35. $f =$

$$\frac{\cos A\cos B\cos C-\sin A\sin B\cos C-\sin A\cos B\sin C-\cos A\sin B\sin C}{\sin A\cos B\cos C+\cos A\sin B\cos C+\cos A\cos B\sin C-\sin A\sin B\sin C}$$

$$= \frac{\cot A\cot B\cot C-\cot C-\cot B-\cot A}{\cot B\cot C+\cot A\cot C+\cot A\cot B-1}$$

(by dividing numerator and denominator by $\sin A\sin B\sin C$).

36. $$f = \tan(2A+A) = \frac{\tan 2A+\tan A}{1-\tan 2A\tan A} = \frac{\dfrac{2\tan A}{1-\tan^2 A}+\tan A}{1-\dfrac{2\tan A\tan A}{1-\tan^2 A}} = l.$$

Let $A = 30^\circ$, then $\tan 3A = \infty$, and $1-3\tan^2 A = 0$, $\therefore \tan A = \dfrac{1}{\sqrt{3}}$.

Let $A = 60^\circ$, then $\tan 3A = 0$, and $\tan A\,(3-\tan^2 A) = 0$,

$\therefore\ 3-\tan^2 A = 0$, and $\tan A = \sqrt{3}$.

37. $$f = \frac{\cos A\cos B-\sin A\sin B}{\sin A\cos B+\cos A\sin B} = \frac{1-\tan A\tan B}{\tan A+\tan B}$$

(by dividing numerator and denominator by $\cos A\cos B$).

Hence $$\cot 75^\circ = \cot(45^\circ+30^\circ) = \frac{1-\tan 45^\circ\tan 30^\circ}{\tan 45^\circ+\tan 30^\circ} = \frac{1-\dfrac{1}{\sqrt{3}}}{1+\dfrac{1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}.$$

38. $$f = \frac{1}{\cos(A+B)} = \frac{1}{\cos A\cos B-\sin A\sin B} = \frac{\sec A\sec B}{1-\tan A\tan B}$$

$$= \frac{\sec A\sec B}{1-\sqrt{[(\sec^2 A-1)(\sec^2 B-1)]}}.$$

39. $\sin 2A = \frac{1}{2}$, $\therefore$ $$\tan A = \frac{\sin A}{\cos A} = \frac{2\sin A\cos A}{2\cos^2 A} = \frac{\sin 2A}{1+\cos 2A}$$

$$= \frac{\sin 2A}{1+\sqrt{(1-\sin^2 2A)}} = \tfrac{1}{2} \div [1+\sqrt{(1-\tfrac{1}{4})}] = \tfrac{1}{2} \div (1+\tfrac{1}{2}\sqrt{3})$$

$$= \tfrac{1}{2}\times\frac{2}{2+\sqrt{3}} = 2-\sqrt{3}.$$

40. $\sin 54^\circ = \cos(90^\circ - 54^\circ) = \cos 36^\circ = 1 - 2\sin^2 18^\circ$
$$= 1 - 2\,\frac{6-2\sqrt{5}}{16} = \frac{8-6+2\sqrt{5}}{8} = \frac{2(\sqrt{5}+1)}{8} = \frac{\sqrt{5}+1}{4}.$$

41. $1 - 2\sin^2 36^\circ = \cos 72^\circ = \dfrac{\sqrt{5}-1}{4}$, $\therefore\ 2\sin^2 36^\circ = 1 - \dfrac{\sqrt{5}-1}{4} =$
$\dfrac{5-\sqrt{5}}{4}$, $\therefore\ \sin^2 36^\circ = \dfrac{5-\sqrt{5}}{8}$, $\therefore\ \sin 36^\circ = \dfrac{\sqrt{(5-\sqrt{5})}}{2\sqrt{2}}$.

42. $\sin(A-B) = \sin[A+(-B)] = \sin A\cos(-B) + \cos A\sin(-B)$
$= \sin A\cos B - \cos A\sin B$.

43. $\sin(A-B) = -\cos[(90^\circ+A)-B]$
$= -[\cos(90^\circ+A)\cos B + \sin(90^\circ+A)\sin B] = \sin A\cos B - \cos A\sin B$.
Let $A = B$, then $1 = \cos(A-A) = \cos A\cos A + \sin A\sin A = \cos^2 A + \sin^2 A$.

44. $A + B < 90^\circ$, $\therefore\ A < 90^\circ - B$, $\therefore\ \cot A > \cot(90^\circ - B) > \tan B$
$> \dfrac{1}{\cot B}$, $\therefore\ \cot A\cot B > 1$.

EXERCISE VII.

2. (i.) $f = \sin(\theta+\phi+\theta-\phi) - \sin(\theta+\phi-\theta+\phi) = l$.
(ii.) $f = \sin(A+2B+A-2B) + \sin(A+2B-A+2B) = l$.
(iii.) $f = \cos(2\alpha+\beta+\alpha+2\beta) + \cos(2\alpha+\beta-\alpha-2\beta) = l$.
(iv.) $f = \cos(4A-3A) - \cos(4A+3A) = l$.
(v.) $f = \sin(\frac{1}{4}\pi + A + \frac{1}{4}\pi - A) + \sin(\frac{1}{4}\pi + A - \frac{1}{4}\pi + A)$
$= \sin\frac{1}{2}\pi + \sin 2A = l$.
(vi.) $f = \cos(\frac{1}{2}\pi + \frac{1}{2}A + \frac{1}{2}\pi - \frac{1}{2}A) + \cos(\frac{1}{2}\pi + \frac{1}{2}A - \frac{1}{2}\pi + \frac{1}{2}A)$
$= \cos\pi + \cos A = l$.
(vii.) $f = 2[\cos(60^\circ + A - 60^\circ + A) - \cos(60^\circ + A + 60^\circ - A)]$
$= 2(\cos 2A - \cos 120^\circ) = l$.
(viii.) $f = \cos(\frac{5}{12}\theta - \frac{1}{12}\theta) - \cos(\frac{5}{12}\theta + \frac{1}{12}\theta) = l$.

4. (i.) $f = 2\sin\frac{1}{2}(30^\circ + A + 30^\circ - A)\sin\frac{1}{2}(30^\circ + A - 30^\circ + A)$
$= 2\sin 30^\circ\sin A = l$.
(ii.) $f = 2\sin\frac{1}{2}(60^\circ + A + 60^\circ - A)\cos\frac{1}{2}(60^\circ + A - 60^\circ + A)$
$= 2\sin 60^\circ\cos A = l$.
(iii.) $f = 2\cos\frac{1}{2}(\frac{1}{4}\pi + \theta + \frac{1}{4}\pi - \theta)\sin\frac{1}{2}(\frac{1}{4}\pi + \theta - \frac{1}{4}\pi + \theta)$
$= 2\cos\frac{1}{4}\pi\sin\theta = l$.
(iv.) $f = \dfrac{2\cos\frac{1}{2}(2\alpha - 3\beta + 5\beta)\cos\frac{1}{2}(2\alpha - 3\beta - 5\beta)}{2\sin\frac{1}{2}(2\alpha - 3\beta + 5\beta)\cos\frac{1}{2}(2\alpha - 3\beta - 5\beta)} = \dfrac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)} = l$.

CHAPTER XI.

1. $f = 2\cos 60^\circ\cos A = 2 \times \frac{1}{2}\cos A = l$. **2.** $l = 2\sin\frac{3}{2}A\cos\frac{1}{2}A =$ $\sin\frac{4}{2}A + \sin\frac{2}{2}A = f$. **3.** $f = 2(\cos 60^\circ + \cos 2A) = 2(\frac{1}{2} + \cos 2A) = l$.
4. $f = 2\cos 2B\sin A = l$. **5.** $l = 2\sin 3A\cos A - \sin 3A =$ $\sin 4A + \sin 2A - \sin 3A = f$. **6.** $l = 2\cos\frac{3}{2}\theta\cos\frac{1}{2}\theta + \cos\frac{3}{2}\theta = f$.

7. $f = \dfrac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)} = \dfrac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A+B)} = l.$

8. $f = \dfrac{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}{2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)} = \dfrac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A-B)} = l.$

9. $f = \dfrac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)} = l.$

10. $f = \dfrac{2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}{2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)} = l.$ 11. $f = \dfrac{2\sin 2A\sin A}{2\cos 2A\sin A} = l.$

12. $f = \dfrac{2\sin A\cos B}{-2\sin A\sin B} = l.$ 13. $f = \dfrac{2\sin\frac{3}{2}\theta\cos\frac{1}{2}\theta}{2\sin\frac{3}{2}\theta\sin\frac{1}{2}\theta} = l.$

14. $l = \cos[(\alpha+\beta)+\alpha]\cos[(\alpha+\beta)-\alpha] = f.$ (See p. 60, Ex. 22.)

15. $f = \sin(3A+2A)\sin(3A-2A) = l.$ (See p. 60, Ex. 21.)

16. $f = \dfrac{\cos^2\beta-\cos^2\alpha}{\cos^2\alpha\cos^2\beta} = \dfrac{1}{\cos^2\alpha} - \dfrac{1}{\cos^2\beta} = \sec^2\alpha - \sec^2\beta$
$= \tan^2\alpha + 1 - (\tan^2\beta + 1) = l.$

17. $f = \dfrac{2\sin 3\alpha\sin\alpha}{2\cos 3\alpha\sin\alpha} = \dfrac{\sin 3\alpha}{\cos 3\alpha} = l.$

18. $f = \dfrac{2\sin\frac{3}{2}A\cos\frac{1}{2}A}{2\cos\frac{3}{2}A\cos\frac{1}{2}A} = \dfrac{\sin\frac{3}{2}A}{\cos\frac{3}{2}A} = l.$

19. $f = \dfrac{\sin\alpha+\sin 5\alpha+2\sin 3\alpha}{\sin 3\alpha+\sin 7\alpha+2\sin 5\alpha} = \dfrac{2\sin 3\alpha\cos 2\alpha+2\sin 3\alpha}{2\sin 5\alpha\cos 2\alpha+2\sin 5\alpha}$
$= \dfrac{2\sin 3\alpha(\cos 2\alpha+1)}{2\sin 5\alpha(\cos 2\alpha+1)} = l.$

20. $f = \dfrac{\sin 2\theta}{\cos 2\theta} - \dfrac{\sin\theta}{\cos\theta} = \dfrac{\sin 2\theta\cos\theta - \cos 2\theta\sin\theta}{\cos 2\theta\cos\theta} = \dfrac{2\sin(2\theta-\theta)}{2\cos 2\theta\cos\theta} = l.$

21. $f = \dfrac{\sin 2A}{\cos 2A} + \dfrac{\sin A}{\cos A} = \dfrac{\sin 2A\cos A + \cos 2A\sin A}{\cos 2A\cos A} = \dfrac{2\sin(2A+A)}{2\cos 2A\cos A} = l.$

22. Transposing, we have $\sin(A+C)\cos C - \cos(A+C)\sin C =$ $\sin(A+B)\cos B - \cos(A+B)\sin B$, or $\sin[(A+B)-B] =$ $\sin[(A+C)-C]$, which is evidently true.

23. $2f = \sin 2A - \sin 2B + \sin 2B - \sin 2C + \sin 2C - \sin 2D + \sin 2D$
$- \sin 2A = 0.$

24. $l = 2\sin A\,.\,2\sin B\sin C = 2\sin A[\cos(B-C) - \cos(B+C)] =$
$2\sin A\cos(B-C) - 2\sin A\cos(B+C) = f.$

25. $l = 2\cos A[\cos(B+C) - \cos(B-C)] = 2\cos A\cos(B+C) +$
$2\cos A\cos(B-C) = f.$

26. $f = 2\cos A[\cos 240^\circ + \cos 2A] = 2\cos A \times (-\frac{1}{2}) + 2\cos A\cos 2A$
$= -\cos A + \cos 3A + \cos A = l.$

27. $l = 2\,.\,2\cos A\cos 3A\,.\,2\cos^2 3A = 2(\cos 4A + \cos 2A)(1+\cos 6A)$
$= 2\cos 4A + 2\cos 2A + 2\cos 4A\cos 6A + 2\cos 2A\cos 6A = f.$

28. $f = \cos\theta + 2\cos\frac{2}{3}\pi\cos\theta = \cos\theta + 2(-\frac{1}{2})\cos\theta = 0.$

29. $f = 2\sin\theta(\cos 2\theta - \cos\frac{2}{3}\pi) = 2\sin\theta\cos\theta - 2\sin\theta\times(-\frac{1}{2})$
$= \sin 2\theta - \sin\theta + \sin\theta = l.$

30. $f = 2\cos^2\theta(\sin 4\theta + \sin 2\theta) + 2\sin^2\theta(\sin 4\theta - \sin 2\theta) = 2\sin 2\theta \times$
$(\cos^2\theta - \sin^2\theta) + 2\sin 4\theta(\cos^2\theta + \sin^2\theta) = 2\sin 2\theta\cos 2\theta + 2\sin 4\theta = l.$

31. $2f = \sin^2\theta(\cos 2\theta - \cos 4\theta) + \cos^2\theta(\cos 2\theta + \cos 4\theta)$
$= \cos 2\theta(\cos^2\theta + \sin^2\theta) + \cos 4\theta(\cos^2\theta - \sin^2\theta)$
$= \cos 2\theta + (2\cos^2 2\theta - 1)\cos 2\theta = 2l.$

32. Remembering that $A = 180° - (B + C)$,
$l = 2\sin A \,.\, 2\sin B\sin C = 2\sin A[\cos(B-C) - \cos(B+C)]$
$= 2\sin(B+C)\cos(B-C) + 2\sin A\cos A$
$= \sin 2B + \sin 2C + \sin 2A = f.$

33. Remembering $\frac{1}{2}A = 90° - \frac{1}{2}(B+C)$,
$l = 1 + 2\sin\frac{1}{2}A[\cos\frac{1}{2}(B-C) - \cos\frac{1}{2}(B+C)]$
$= 1 + 2\cos\frac{1}{2}(B+C)\cos\frac{1}{2}(B-C) - 2\sin^2\frac{1}{2}A = f.$

34. $l = 2\sin\frac{1}{2}A[\sin\frac{1}{2}(B+C) - \sin\frac{1}{2}(B-C)]$
$= 2\sin\frac{1}{2}A\cos\frac{1}{2}A - 2\cos\frac{1}{2}(B+C)\sin\frac{1}{2}(B-C) = f.$

35. $\because \pi - A = B + C$, &c., $\therefore l = 2\cos\frac{1}{2}(\pi - A)\, 2\cos\frac{1}{2}(A+C)\times$
$\cos\frac{1}{2}(A+B) = 2\cos\frac{1}{2}(\pi - A)[\cos\frac{1}{2}(A+A+B+C) + \cos\frac{1}{2}(B-C)]$
$= 2\cos\frac{1}{2}(\pi - A)\cos\frac{1}{2}(\pi + A) + 2\cos\frac{1}{2}(B+C)\cos\frac{1}{2}(B-C)$
$= \cos\frac{1}{2}\pi + \cos\frac{1}{2}A + \cos\frac{1}{2}B + \cos\frac{1}{2}C = f.$

36. $\cos A \,.\, 2\cos B\cos C = \cos A[\cos(B+C) + \cos(B-C)]$
$= -\cos^2 A - \cos(B+C)\cos(B-C) = -\cos^2 A - (\cos^2 B - \sin^2 C)$
$= -\cos^2 A - \cos^2 B - \cos^2 C + 1$; $\therefore$ transposing, $f = l.$

37. $f = \dfrac{\sin A + \sin B - \sin(A+B)}{\sin A + \sin B + \sin(A+B)}$
$= \dfrac{2[\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B) - \sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A+B)]}{2[\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B) + \sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A+B)]}$
$= \dfrac{\cos\frac{1}{2}(A-B) - \cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B) + \cos\frac{1}{2}(A+B)} = \dfrac{2\sin\frac{1}{2}A\sin\frac{1}{2}B}{2\cos\frac{1}{2}A\cos\frac{1}{2}B} = \tan\frac{1}{2}A\tan\frac{1}{2}B.$

38. $\because A + B = 90° - C$, $\therefore \tan(A+B) = \cot C$,
$\therefore \dfrac{\tan A + \tan B}{1 - \tan A\tan B} = \dfrac{1}{\tan C}$, $\therefore \tan A\tan C + \tan B\tan C = 1 - \tan A\tan B$;
transposing, $f = l.$

39. $1 = \sin(A+B+C) = \sin A\cos B\cos C + \cos A\sin B\cos C +$
$\cos A\cos B\sin C - \sin A\sin B\sin C$; dividing by $\cos A\cos B\cos C$,
$\sec A\sec B\sec C = \tan A + \tan B + \tan C - \tan A\tan B\tan C$, $\therefore f = l.$

40. $f = \dfrac{2\cos^2 A - \sin^2 A}{\cos A + \sin A} + \dfrac{2\sin 2A}{\cos A - \sin A} = \dfrac{2(\cos A - \sin A)}{1} + \dfrac{2\sin 2A}{\cos A - \sin A}$
$= 2\,\dfrac{\cos^2 A + \sin^2 A - 2\sin A\cos A + \sin 2A}{\cos A - \sin A} = \dfrac{2}{\cos A - \sin A}$
$= \dfrac{\sqrt{2}}{\frac{1}{\sqrt{2}}\cos A - \frac{1}{\sqrt{2}}\sin A} = l.$

41. $\because \beta + \gamma = \pi - \alpha$, $\therefore f = (1 - \cos\alpha)(1 + \cos\alpha) = \sin^2\alpha = \sin\alpha\sin(\beta + \gamma)$

42. $\because B+C=180°-A$, $\therefore \tan A=-\tan(B+C)=-\dfrac{\tan B+\tan C}{1-\tan B\tan C}$,
$\therefore \tan A-\tan A\tan B\tan C=-\tan B-\tan C$, $\therefore f=l$. Suppose the $\triangle$ equilateral, $\because A=B=C=60°$, $\therefore 3\tan A=\tan^3 A$, $\therefore \tan^2 A=3$, and $\tan A=\sqrt{3}$.

43. $\because \dfrac{a}{b}=\dfrac{\cos\alpha}{\cos\beta}$, $\therefore \dfrac{a+b}{a-b}=\dfrac{\cos\alpha+\cos\beta}{\cos\alpha-\cos\beta}=\dfrac{2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\beta-\alpha)}{2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\beta-\alpha)}=l.$

44. $a\cos 2\theta+b\sin 2\theta=a(1-2\sin^2\theta)+b\,.\,2\sin\theta\cos\theta$
$=a-2(a\sin^2\theta-b\sin\theta\cos\theta)=a-2\cos^2\theta\,(a\tan^2\theta-b\tan\theta)$
$=a-2\cos^2\theta\left(a\dfrac{b^2}{a^2}-b\dfrac{b}{a}\right)=a-2\cos^2\theta\times 0=a.$

45. (1) $\sin A+\sin B=2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$; in which $\sin\frac{1}{2}(A+B)$ is constant, and $\cos\frac{1}{2}(A-B)$ is greatest when $\frac{1}{2}(A-B)=0$, *i.e.*, when $A=B$. (2) $\tan A+\tan B=\dfrac{\sin(A+B)}{\cos A\cos B}$ [p. 58, (1)] $=\dfrac{2\sin(A+B)}{\cos(A+B)+\cos(A-B)}$. Now numerator is constant, and in denominator $\cos(A-B)$ alone varies, and is *greatest* (and $\therefore$ the fraction is least) when $A-B=0$; *i.e.*, when $A=B$.

EXERCISE VIII.

2. $f=b\sin A-b\sin A=l$. 3. $f=a\sin B+a\sin C=l$.
4. $f=a\sin B-b\sin A+b\sin C-c\sin B+c\sin A-a\sin C=0$, by (2).
5. $f=b\sin A-a\sin B+c\sin B-b\sin C=0$, by (2).
6. $l^2=\dfrac{1-\cos^2 A}{1-\cos^2 B}=\dfrac{\sin^2 A}{\sin^2 B}=f^2$; $\therefore l=f$.
7. $f=\dfrac{a}{\cos A}+\dfrac{b}{\cos B}=\dfrac{a\cos B+b\cos A}{\cos A\cos B}=\dfrac{c}{\cos A\cos B}=l.$
8. $f=(b\cos C+c\cos B)+(c\cos A+a\cos C)+(a\cos B+b\cos A)=l.$
9. $f=\dfrac{\cos B}{\sin B}+\dfrac{\cos A}{\sin A}=\dfrac{\sin A\cos B+\cos A\sin B}{\sin A\sin B}=\dfrac{\sin(A+B)}{\sin A\sin B}$
$=\dfrac{\sin C}{\sin A\sin B}=\dfrac{c}{b\sin A}=l.$
10. $l=a^2+b^2-c^2+c^2+a^2-b^2+b^2+c^2-a^2=f.$
11. $l=-\dfrac{b^2+c^2-a^2}{2bc}=-\cos A=f.$

CHAPTER XII.

1. $b=a\dfrac{\sin B}{\sin A}=\sqrt{6}\dfrac{\sin 45°}{\sin 60°}=\sqrt{6}\times\dfrac{1}{\sqrt{2}}\times\dfrac{2}{\sqrt{3}}=2.$

2. $\sin A=\dfrac{a}{b}\sin B=\dfrac{3\sqrt{2}}{2\sqrt{3}}\sin 45°=\dfrac{\sqrt{3}}{\sqrt{2}}\times\dfrac{1}{\sqrt{2}}=\dfrac{\sqrt{3}}{2}$; $\therefore A=60°$ or $120°$.
The solution is ambiguous, $\because a>b$.

3. $\sin B = \frac{b}{a}\sin A = \frac{2}{\sqrt{2}}\sin 30^\circ = \sqrt{2}\times\frac{1}{2} = \frac{1}{\sqrt{2}}$; $\therefore B = 45^\circ$ or 135°, $\because b > a$.

4. $\sin B = \frac{b}{a}\sin A = \frac{\sqrt{2}}{2}\sin 45^\circ = \frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}} = \frac{1}{2}$; $\therefore B = 30^\circ$ (only, $\because b < a$).

5. $a^2 = 3^2 + 5^2 - 2\,.\,3\,.\,5\,.\cos 120^\circ = 9 + 25 + 30\times\frac{1}{2} = 49$, $\because a = 7$.

6. $\cos C = [(\frac{6}{5})^2 + (\frac{8}{5})^2 - 2^2] \div (2\times\frac{6}{5}\times\frac{8}{5}) = (\frac{36}{25} + \frac{64}{25} - 4)\times\frac{25}{96} = 0$;
$\therefore C = 90^\circ$.

7. $\cos A = \frac{(2\sqrt{7})^2 + (3\sqrt{7})^2 - 7^2}{2\,.\,2\sqrt{7}\,.\,3\sqrt{7}} = \frac{28 + 63 - 49}{84} = \frac{42}{84} = \frac{1}{2}$, $\therefore A = 60^\circ$.

8. $\sin B = \frac{b}{a}\sin A = \frac{5}{2}\times\frac{1}{2} = \frac{5}{4}$, which is impossible. Hence no triangle can have the parts "given."

9. $\sin B = \frac{b}{a}\sin A = \frac{\sqrt{20}+2}{2}\sin 18^\circ = \frac{2(\sqrt{5}+1)}{2}\times\frac{\sqrt{5}-1}{4} = \frac{5-1}{4}$
$= 1$, $\therefore B = 90^\circ$; $\therefore C = 90^\circ - 18^\circ = 72^\circ$; and $c = \sqrt{(b^2 - a^2)} = \sqrt{(24 + 4\sqrt{20} - 4)} = \sqrt{(20 + 8\sqrt{5})} = 2\sqrt{(5 + 2\sqrt{5})}$. 90° and its supplement are identical.

10. Suppose a, b, C the given sides and angle; then A, B, c are required.
$$c^2 = (3\sqrt{6})^2 + 3^2(\sqrt{3}+1)^2 - 2\,.\,3\sqrt{6}\,.\,3(\sqrt{3}+1)\frac{1}{\sqrt{2}}$$
$$= 54 + 9(4 + 2\sqrt{3}) - 18\sqrt{3}(\sqrt{3}+1) = 54 + 36 - 54 + 18\sqrt{3} - 18\sqrt{3}$$
$$= 36;\ \therefore c = 6.\quad \therefore \sin A = \frac{a}{c}\sin C = \frac{3\sqrt{6}}{6}\,\frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2};$$
$\therefore A = 60^\circ$; $\therefore B = 180^\circ - (A + C) = 180^\circ - 105^\circ = 75^\circ$.

11. C = greatest and A = least angle. $\cos C = \frac{3^2 + 4^2 - 5^2}{2\,.\,3\,.\,4} = 0$,
$\therefore C = 90^\circ$; $\therefore \cos A = \frac{4}{5} = \cdot 8$, $\therefore A = 36^\circ\,52'$.

12. Let B be the angle, $\cos B = \frac{5^2 + 9^2 - 8^2}{2\,.\,5\,.\,9} = \frac{42}{90} = \cdot 4\dot{6}$,
$\therefore B = 62^\circ\,11'$ nearly.

13. $\cos C = [1^2 + (\frac{6}{5})^2 - 2^2] \div (2\,.\,1\,.\,\frac{6}{5}) = (1 + \frac{36}{25} - 4) \div \frac{12}{5} = -\frac{39}{25}\times\frac{5}{12}$
$= -\frac{13}{20} = -\cdot 65 = -\cos 49^\circ 33' = \cos(180^\circ - 49^\circ 33')$; $\therefore C = 130^\circ 27'$.

14. $c^2 = 5^2 - 8^2 - 2\,.\,5\,.\,8\,.\cos 97^\circ\,53'\,50'' = 89 + 80\cos 82^\circ\,6'\,10''$
$= 89 + 80\times\cdot 1375 = 89 + 11 = 100$; $\therefore c = 10$.

15. $a = b\frac{\sin A}{\sin B} = 5\frac{1}{3}\times\frac{\sin 131^\circ\,35'}{\sin 30^\circ}$ fur. $= \frac{16}{3}\times\frac{\sin 48^\circ 25'}{\sin 30^\circ}$ fur.
$= \frac{16}{3}\times\frac{3}{4}\times\frac{2}{1}$ fur. $= 8$ fur. $= 1$ mile.

16. $\sin A = \frac{a}{b}\sin B = \frac{15}{8}\times\frac{4}{10} = \frac{3}{4} = \cdot 75$; $\therefore A = 48^\circ\,25'$ or $131^\circ\,35'$.

17. If a, b contain C, $\because \sqrt{(a^2+ab+b^2)} >$ either a or b, C is greatest angle. $\cos C = \dfrac{a^2+b^2-a^2-ab-b^2}{2\,.\,a\,.\,b} = -\frac{1}{2}$; $\therefore C = 120°$.

18. $2f = 2ab\cos C - 2ac\cos B = a^2+b^2-c^2-(a^2+c^2-b^2) = 2b^2-2c^2 = 2l$.

19. $f = a^2\sin A + b\,.\,a\sin B + c\,.\,a\sin C = a^2\sin A + b\,.\,b\sin A + c\,.\,c\sin A = l$.

20. $\dfrac{2(\cos B\cos C+\cos A)}{2(\cos A\cos C+\cos B)} = \dfrac{\cos(B+C)+\cos(B-C)-2\cos(B+C)}{\cos(A+C)+\cos(A-C)-2\cos(A+C)}$

$= \dfrac{\cos(B-C)-\cos(B+C)}{\cos(A-C)-\cos(A+C)} = \dfrac{2\sin B\sin C}{2\sin A\sin C} = \dfrac{b}{a}$.

Multiplying out, $2f = 2l$, $\therefore f = l$.

21. $\dfrac{2(\cos B+\cos C\cos A)}{2(\cos C+\cos A\cos B)} = \dfrac{-2\cos(C+A)+\cos(C+A)+\cos(C-A)}{-2\cos(A+B)+\cos(A+B)+\cos(A-B)}$

$= \dfrac{\cos(C-A)-\cos(C+A)}{\cos(A-B)-\cos(A+B)} = \dfrac{2\sin C\sin A}{2\sin A\sin B} = \dfrac{c}{b}$.

Multiplying out, $2f = 2l$.

22. $2ab\sin(C-30°) - 2bc\sin(A-30°)$
$= b\left[2a\left(\sin C\tfrac{1}{2}\sqrt{3} - \cos C\tfrac{1}{2}\right) - 2c\left(\sin A\tfrac{1}{2}\sqrt{3} - \cos A\tfrac{1}{2}\right)\right]$
$= b\left[\sqrt{3}(a\sin C - c\sin A) + c\cos A - a\cos C\right]$
$= \dfrac{2b}{2}\left[\sqrt{3}\times 0 + c\cos A - a\cos C\right] = \frac{1}{2}\left[2bc\cos A - 2ab\cos C\right]$
$= \frac{1}{2}\left[b^2+c^2-a^2-(a^2+b^2-c^2)\right] = c^2-a^2$. By transposition, $f = l$.

23. $f = (a\cos C + c\cos A)\cos B + (a\cos B + b\cos A)\cos C = 2a\cos B\cos C$
$+ (c\cos B + b\cos C)\cos A = a[\cos(B+C)+\cos(B-C)] + a\cos A$
$= -a\cos A + a\cos(B-C) + a\cos A = l$.

24. Add $a\cos A = -a\cos(B+C)$ to 23. Then $f = -a\cos(B+C)$
$+ a\cos(B-C) = a[\cos(B-C)-\cos(B+C)] = 2a\sin B\sin C$.

25. $l = 1 + \dfrac{2\sin A\sin B\sin C}{\sin A+\sin B+\sin C}$

$= 1 + \dfrac{2\,.\,2\sin\frac{1}{2}A\cos\frac{1}{2}A\,.\,2\sin\frac{1}{2}B\cos\frac{1}{2}B\,.\,2\sin\frac{1}{2}C\cos\frac{1}{2}C}{4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}$ [p. 64, (4)].

$= 1 + 4\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C = f$ (p. 66, 33).

26. $A+B+C = 180°$, $\therefore A+B = 180°-C$, $\therefore \frac{1}{2}(A+B) = 90°-\frac{1}{2}C$,
$\therefore f = l$.

27. Remembering that $\frac{1}{2}(A+B) = 90°-\frac{1}{2}C$, $\dfrac{a-b}{c} = \dfrac{\sin A-\sin B}{\sin C}$

$= \dfrac{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}{2\sin\frac{1}{2}C\cos\frac{1}{2}C} = \dfrac{\sin\frac{1}{2}C\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}C\cos\frac{1}{2}C} = \dfrac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}$.

28. $\dfrac{a+b}{c} = \dfrac{\sin A+\sin B}{\sin C} = \dfrac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\sin\frac{1}{2}C\cos\frac{1}{2}C} = \dfrac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}$.

$\therefore f = l$.

29. $l = \dfrac{\sin B+\sin C}{\sin A} = \dfrac{2\sin\frac{1}{2}(B+C)\cos\frac{1}{2}(B-C)}{2\sin\frac{1}{2}A\cos\frac{1}{2}A} = \dfrac{\cos\frac{1}{2}(B-C)}{\sin\frac{1}{2}A}$

$$= \frac{\cos\frac{1}{2}(C-B)}{\sin^{\frac{1}{2}} A} = f, \text{ for } \cos\tfrac{1}{2}(C-B) = \sin(90^\circ - \tfrac{1}{2}C + \tfrac{1}{2}B)$$
$$= \sin(\tfrac{1}{2}A + \tfrac{1}{2}B + \tfrac{1}{2}C - \tfrac{1}{2}C + \tfrac{1}{2}B).$$

30. Let $A = 2a$, then $B = 3a$, $C = 4a$,

$$\therefore\ l = \frac{\sin 2a + \sin 4a}{2\sin 3a} = \frac{2\sin 3a\cos a}{2\sin 3a} = \cos a = f.$$

31. $l = \dfrac{b\,.\,b\sin C + c\,.\,c\sin B}{b+c} = \dfrac{b\sin C\,(b+c)}{b+c} = b\sin C = f.$

(See fig. to Art. 20.)

32. $\dfrac{a^2-b^2}{c^2} = \dfrac{\sin^2 A - \sin^2 B}{\sin^2 C} = \dfrac{\sin(A+B)\sin(A-B)}{\sin^2(A+B)}$

$= \dfrac{\sin(A-B)}{\sin(A+B)} = \dfrac{\sin(A-B)}{\sin C}, \quad \therefore\ f = l.$

33. $2\sin\frac{1}{2}(A-B)\,.\,2\sin\frac{1}{2}(B-C)\sin\frac{1}{2}(C-A)$

$= 2\sin\frac{1}{2}(A-B)\left[\cos\frac{1}{2}(A+B-2C) - \cos\frac{1}{2}(A-B)\right]$

$= -2\sin\frac{1}{2}(B-A)\cos\frac{1}{2}(A+B-2C) - 2\sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A-B)$

$= -\sin(B-C) - \sin(C-A) - \sin(A-B) = -\dfrac{b^2-c^2}{a^2}\sin A$

$-\dfrac{c^2-a^2}{b^2}\sin B - \dfrac{a^2-b^2}{c^2}\sin C$, by 32. By transposition, $f = 0$.

CHAPTER XIII.

1. Let $x = \log_2 256$; then $2^x = 256 = 2^8$; $\therefore\ x = 8$.

Let $x = \log_{2\sqrt{2}} 256$; then $(2\sqrt{2})^x = 256$, or $(2^{\frac{3}{2}})^x = 2^8$, or $2^{\frac{3}{2}x} = 2^8$;

$\therefore\ \frac{3}{2}x = 8$, $\therefore\ x = \frac{16}{3} = 5\cdot\dot{3}$.

2. Let $x = \log_{10} 100$; then $10^x = 100 = 10^2$; $\therefore\ x = 2$.

Let $x = \log_{10}\cdot 001$; then $10^x = \cdot 001 = \dfrac{1}{10^3} = 10^{-3}$; $\therefore\ x = -3$.

Let $x = \log_{\cdot 01} 1000$; then $(\cdot 01)^x = 1000$, or $\left(\dfrac{1}{10^2}\right)^x = 10^3$, or

$10^{-2x} = 10^3$; $\therefore\ -2x = 3$, $\therefore\ x = -\frac{3}{2} = -1\cdot 5$.

3. Let $x = \log_{3\sqrt{3}} 81\sqrt[3]{3}$; then $(3\sqrt{3})^x = 81\sqrt[3]{3}$, or $(3^{\frac{3}{2}})^x = 3^4\,.\,3^{\frac{1}{3}}$,

or $3^{\frac{3}{2}x} = 3^{\frac{13}{3}}$; $\therefore\ \frac{3}{2}x = \frac{13}{3}$; $\therefore\ x = \frac{26}{9} = 2\cdot\dot{8}$.

4. Let $x = \log a^{\frac{p}{q}}$ to base $a^{\frac{m}{n}}$; then $(a^{\frac{m}{n}})^x = a^{\frac{p}{q}}$; $\therefore\ a^{\frac{mx}{n}} = a^{\frac{p}{q}}$,

$\therefore\ \dfrac{mx}{n} = \dfrac{p}{q}$, $\therefore\ x = \dfrac{np}{mq}$.

5. Let x = required no.; then $-5 = \log_{10} x$, $\therefore\ x = 10^{-5} = \cdot 00001$.

6. Let x = required base; then $2 = \log_x 10$; $\therefore\ x^2 = 10$,

$\therefore\ x = \sqrt{10} = 3\cdot 162$.

7. Let $x = \log_2 N$, and $y = \log_8 N$; then $N = 2^x$, and $N = 8^y =$
$(2^3)^y = 2^{3y}$, $\therefore\ 2^x = 2^{3y}$; $\therefore\ x = 3y$.

8. Let $x = \log_{25} 8$; then $25^x = 8$ or $5^{2x} = 2^3$; $\therefore\ 2x \log 5 = 3 \log 2$,

$$\therefore\ x = \frac{3\log 2}{2\log 5} = \frac{3\log 2}{2(1-\log 2)} = \frac{\cdot 90309}{1\cdot 39794} = \cdot 646.$$

In 9 – 22, g stands for given logarithm.

9. $g = \log x + \log y + \log z$.

10. $g = \log(10\times 2) = \log 10 + \log 2 = 1 + \log 2$.

11. $g = \log(3\times 5\times 10) = \log 3 + \log 5 + \log 10 = \log 3 + 1 - \log 2 + 1$
$= 2 - \log 2 + \log 3$.

12. $g = \log 1 - \log 2^2 = -2\log 2$.

13. $g = \log(ab^2c^3) - \log(2^2x^4y^5)$
$= \log a + 2\log b + 3\log c - 2\log 2 - 4\log x - 5\log y$.

14. $g = \log(2\times 3ab^{\frac{3}{2}}c^{\frac{1}{3}}d^{\frac{1}{3}}) = \log 2 + \log 3 + \log a + \frac{3}{2}\log b + \frac{1}{3}\log c + \frac{1}{3}\log d$.

15. $g = \log\dfrac{(2^4\times 3)^{\frac{1}{4}}}{3^4} = \log\dfrac{2\,.\,3^{\frac{1}{4}}}{3^4} = \log\dfrac{2}{3^{\frac{15}{4}}} = \log 2 - \frac{15}{4}\log 3$.

16. $g = \log\dfrac{\sqrt[3]{3^2}\times\sqrt[6]{2^3}}{\sqrt[4]{5}\times\sqrt[20]{2^2}} = \log\dfrac{3^{\frac{2}{3}}\,.\,2^{\frac{1}{2}}}{5^{\frac{1}{4}}\,.\,2^{\frac{1}{10}}} = \log\dfrac{3^{\frac{2}{3}}\,.\,2^{\frac{2}{5}}}{5^{\frac{1}{4}}}$.

$= \log(3^{\frac{2}{3}}\,.\,2^{\frac{2}{5}}) - \frac{1}{4}(1-\log 2) = \frac{2}{5}\log 2 + \frac{1}{4}\log 2 + \frac{2}{3}\log 3 - \frac{1}{4}$
$= \frac{13}{20}\log 2 + \frac{2}{3}\log 3 - \cdot 25$.

17. $g = \log(a+b) + \log\sin\frac{1}{2}C + \log\sec\phi$.

18. $g = \log c + \log\sin B - \log\sin C$.

19. $g = \log(ac\sin B) - \log 2 = \log a + \log c + \log\sin B - \log 2$.

20. $g = \log(a-b) + \log\cot\frac{1}{2}C - \log(a+b)$.

21. $g = \log[(b+c)^{\frac{1}{2}}(b-c)^{\frac{1}{2}} = \frac{1}{2}\log(b+c) + \frac{1}{2}\log(b-c)$.

22. $g = \log[(s-b)^{\frac{1}{2}}(s-c)^{\frac{1}{2}}] - \log(b^{\frac{1}{2}}c^{\frac{1}{2}})$
$= \frac{1}{2}\log(s-b) + \frac{1}{2}\log(s-c) - \frac{1}{2}\log b - \frac{1}{2}\log c$.

23. $2^{3x} = 10^2$, $\therefore\ 3x\log 2 = 2\log 10$, $\therefore\ x = \dfrac{2\log 10}{3\log 2} = \dfrac{2}{\cdot 90309} = 2\cdot 214$.

24. $(x-1)\log 5 = \log 2^2$, $\therefore\ x - 1 = \dfrac{2\log 2}{\log 5} = \dfrac{2\log 2}{1-\log 2} = \dfrac{\cdot 60206}{\cdot 69897}$
$= \cdot 862$; $\therefore\ x = 1\cdot 862$.

25. $(x-2)\log(10\times 2) = \log\dfrac{1}{2^2}$, $\therefore\ x - 2 = \dfrac{\log 1 - 2\log 2}{\log 10 + \log 2} = \dfrac{-2\log 2}{1+\log 2}$
$= -\dfrac{\cdot 60206}{1\cdot 30103} = -\cdot 462$; $\therefore\ x = 1\cdot 538$.

26. $2^{x+3} = 25^{3-2x} = 5^{2(3-2x)}$, $\therefore\ (x+3)\log 2 = 2(3-2x)\log 5 =$
$(6-4x)(1-\log 2)$; $\therefore\ x(4-3\log 2) = 6 - 9\log 2$, or $x(4 - \cdot 90309)$
$= 6 - 2\cdot 70927$, or $x\times 3\cdot 09691 = 3\cdot 29073$; $\therefore\ x = 1\cdot 06$.

27. $x(\log 7 - \log 3) = \log 10^2 = 2$, $\therefore\ x = \dfrac{2}{\log 7 - \log 3} = \dfrac{2}{\cdot 36798} = 5\cdot 435$.

28. $2^{3x}\,.\,(2\,.\,3)^{3x-5} = 3^{2x-2}\,.\,2^{5x}$, or $2^{6x-5}\,.\,3^{3x-5} = 3^{2x-2}\,.\,2^{2x}$,
$\therefore\ 2^{x-5}\,.\,3^{x-3} = 1$, $\therefore\ (x-5)\log 2 + (x-3)\log 3 = \log 1 = 0$;

$\therefore \quad x(\log 2 + \log 3) = 5 \log 2 + 3 \log 3,$

$$\therefore\ x = \frac{5\log 2 + 3\log 3}{\log 2 + \log 3} = \frac{1·50515 + 1·43136}{·30103 + ·47712} = \frac{2·93651}{·77815} = 3·774.$$

CHAPTER XIV.

1 and **2** answered by inspection.

3. (1) Their significant figures are the same. (2) The first significant figure is equally distant from the units' place and on same side of it. *Otherwise:* (1) $\log 3·56 = \log(·0356 \times 10^2) = 2 + \log ·0356$. (2) ·0356 and ·04 are both between 10^{-2} and 10^{-1}.

4. Answered by inspection.

5.

$$\begin{array}{r} \bar{3}·72045 \\ 6 \\ \hline \overline{14}·32270 \end{array} \qquad \begin{array}{r} \bar{2}·00243 \\ 5 \\ \hline \overline{10}·01215 \end{array} \qquad \begin{array}{r} \bar{1}·98628 \\ 4 \\ \hline \bar{1}·94512 \end{array} \qquad \begin{array}{r} ·76215 \\ 7 \\ \hline 5·33505 \end{array}$$

$\overline{18} + 4 = \overline{14} \qquad \bar{4} + 3 = \bar{1}$

6.

$$\begin{array}{r} \overline{14}·32270 \\ \overline{10}·01215 \\ \bar{1}·94512 \\ 5·33505 \\ \hline \overline{19}·61502 \end{array}$$

7.

$$4\,)\,\overline{\overline{11}·02156}\ \ \bar{3}·25539 \qquad 6\,)\,\overline{2·56821}\ \ ·42804 \qquad 5\,)\,\overline{\bar{1}·00426}\ \ \bar{1}·80085 \qquad 3\,)\,\overline{\overline{33}·82156}\ \ \overline{11}·27385$$

$\overline{11} = \overline{12} + 1 \qquad \bar{1} = \bar{5} + 4$

8.

$$\begin{array}{r} \bar{3}·25539 \\ ·42804 \\ \hline \bar{4}·82735 \end{array} \qquad \begin{array}{r} \bar{1}·80085 \\ \overline{11}·27385 \\ \hline 10·52700 \end{array}$$

$\bar{3} - 1 = \bar{4} \qquad \bar{1} + \overline{11} = \bar{1} + 11 = 10.$

9. $\log 128 = \log 2^7 = 7 . \log 2 = 2·10721. \qquad 125 = 5^3 = 10^3 + 2^3;$
$\therefore\ \log 125 = 3 \log 10 - 3 \log 2 = 3 - ·90309 = 2·09691.$
$2500 = 10^2 \times 5^2 = 10^4 \div 2^2, \ \therefore\ \log 2500 = 4 - 2 \log 2 = 3·39794.$

10. $·0005 = 1 \div 10^3 \times 2, \therefore \log ·0005 = -3 - \log 2 = -3·30103 = \bar{4}·69897.$
$196 = 49 \times 4 = 7^2 \times 2^2, \ \therefore\ \log 196 = 2 \log 7 + 2 \log 2 = 1·690196 + ·60206 = 2·292256. \qquad \log(28 \div 5)^{\frac{3}{4}} = \frac{3}{4} \log(56 \div 10) = \frac{3}{4}(\log 7 + 3 \log 2 - 1) = \frac{3}{4}(·845098 + ·90309 - 1) = \frac{3}{4}$ of $·748188 = ·561141.$

11. $\log \sin 45° = \log 2^{-\frac{1}{2}} = -\frac{1}{2} \log 2 = -·150515 = \bar{1}·849485.$
$\log \tan 60° = \log 3^{\frac{1}{2}} = \frac{1}{2} \log 3 = ·2385607. \qquad \log \sec 30° = \log(2 \div \sqrt{3}) = \log 2 - \frac{1}{2} \log 3 = ·30103 - ·2385607 = ·0624693.$

12. $\log \sqrt[3]{\frac{1}{6}} = \frac{1}{3}(\log 1 - \log 6) = -\frac{1}{3}(\log 2 + \log 3) = -\frac{1}{3}$ of $·7781513 = -·2593838 = \bar{1}·7406162. \qquad \log(5\frac{1}{3})^{-\frac{1}{2}} = -\frac{1}{2} \log(2^4 \div 3) = -\frac{1}{2}(4 \log 2 - \log 3) = -\frac{1}{2}(1·20412 - ·4771213) = -\frac{1}{2}$ of $·7269987 = -·3634994 = \bar{1}·6365006.$

13. $$\frac{(2.7)^3 \times (·81)^{\frac{4}{5}}}{(90)^{\frac{5}{4}}} = \frac{3^9 \times 3^{\frac{16}{5}}}{10^3 \times 10^{\frac{8}{5}} \times 3^{\frac{5}{2}} . 10^{\frac{5}{4}}} = \frac{3^{\frac{97}{10}}}{10^{\frac{117}{20}}};$$

$\therefore$ its $\log = \frac{97}{10} \log 3 - \frac{117}{20} \log 10 = 4·6280766 - 5·85 = \bar{2}·7780766.$

14. $\log \dfrac{\sqrt[3]{2}\sqrt[4]{3}\sqrt[5]{4}}{\sqrt[6]{5}\sqrt[7]{6}} = \log \dfrac{2^{\frac{1}{3}} \,.\, 2^{\frac{1}{6}} \,.\, 3^{\frac{1}{4}} \,.\, 2^{\frac{2}{5}}}{10^{\frac{1}{6}} \,.\, 2^{\frac{1}{7}} \,.\, 3^{\frac{1}{7}}} = \log \dfrac{2^{\frac{53}{70}} \,.\, 3^{\frac{3}{28}}}{10^{\frac{1}{6}}} = \frac{53}{70}\log 2 + \frac{3}{28}\log 3 - \frac{1}{6} = \cdot 227923 + \cdot 05112014 - \cdot 16666666 = \cdot 1123775.$

15. $\log(3^{20} \times 5^5 \div 2^{11}) = \log(3^{20} \times 10^5 \div 2^{16}) = 20\log 3 + 5 - 16\log 2 = 9\cdot 542426 + 5 - 4\cdot 81648 = 9\cdot 725946$, $\therefore$ No. of digits $= 9 + 1 = 10$.

16. $3 = \dfrac{\sqrt{9}\,.\,\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{18}\,.\,\sqrt{5}}{\sqrt{10}} = \dfrac{\sqrt{18}\,.\,\sqrt[4]{25}}{\sqrt{10}}$; $\therefore \log 3 = \frac{1}{2}\log 18 + \frac{1}{4}\log 25 - \frac{1}{2} = \cdot 627636 + \cdot 349485 - \cdot 5 = \cdot 477121.$

$\cdot 16 = \dfrac{2^4}{100} = \dfrac{2^4\,.\,5^4}{10^2\,.\,25^2} = \dfrac{10^2}{25^2}$; $\therefore \log \cdot 16 = 2 - 2\log 25 = 2 - 2\cdot 795880 = \bar{1}\cdot 204120.$ $450 = 18 \times 25$, $\therefore \log 45 = \log 18 + \log 25 = 2\cdot 653212.$

CHAPTER XV.

The logarithms described as "given" are so only as far as regards mantissa.

1.
$$\begin{array}{lr}
\log 17257 = & 4\cdot 2369653 \\
\log 17256 = & 4\cdot 2369401 \\
 & \overline{252} \\
 & \cdot 74 \\
 & \overline{10|08} \\
 & 176|4 \\
\text{pr. pt. for } \cdot 74 = & \overline{186} \\
\log 17256 = & 4\cdot 2369401 \\
\log 17256\cdot 74 = & \overline{4\cdot 2369587} \\
\therefore \log 1725674 = & 6\cdot 2369587
\end{array}$$

2.
$$\begin{array}{rl}
4\cdot 8924675 = & \log 78067 \\
4\cdot 8924619 = & \log 78066 \\
\overline{56} = & \text{diff. for 1,} \\
4\cdot 8924652 = & \log \text{ given} \\
4\cdot 8924619 = & \log 78066 \\
\overline{33} = & \text{pro. part,}
\end{array}$$
$\therefore$ increase to no. $= \frac{33}{56} = \cdot 589$,
no. reqd. $= \cdot 78066589$.

3.
$$\begin{array}{rl}
-1\cdot 6784533 = & \bar{2}\cdot 3215467, \\
4\cdot 3215467 = & \log \text{ given} \\
4\cdot 3215363 = & \log 20967 \\
\overline{104} = & \text{pro. part,}
\end{array}$$
and $\frac{104}{207} = \cdot 5$,
$\therefore$ no. reqd. $= \cdot 0209675$.

4.
$$\begin{array}{rl}
4\cdot 2183779 = & \log 16534 \\
4\cdot 2183517 = & \log 16533 \\
\overline{262} = & \text{diff. for 1,} \\
4\cdot 2183736 = & \text{given log} \\
4\cdot 2183517 = & \log 16533 \\
\overline{219} = & \text{pro. part,}
\end{array}$$
$\frac{219}{262} = \cdot 8359$,
no. reqd. $= \cdot 00165338359$.

5. Let $x = (\cdot 0000083825)^{\frac{1}{3}}$
$$\begin{array}{rl}
\log x = & \frac{1}{3}\log \cdot 0000083825 \\
= & \frac{1}{3}(\bar{6}\cdot 9233736) \\
= & \bar{2}\cdot 3077912, \\
\log 20313 = & 4\cdot 3077741 \\
\text{pro. part} = & \overline{171}; \\
\log 20314 = & 4\cdot 3077954 \\
\log 20313 = & 4\cdot 3077741 \\
\text{diff. for 1} = & \overline{213},
\end{array}$$
$\frac{171}{213} = \cdot 8$,
$\therefore\ x = \cdot 0203138$.

6.
$$\begin{array}{rl}
\log 105 = & \log(\frac{10}{2} \times 3 \times 7) \\
= & 1 - \log 2 + \log 3 + \log 7 \\
= & \cdot 69897 + \cdot 47712 + \cdot 84510 \\
= & 2\cdot 02119, \\
\therefore \log(1\cdot 05)^{20} = & 20\log 1\cdot 05 \\
= & \cdot 02119 \times 20 \\
= & .42380 \\
\text{and } \log 2\cdot 653 = & \cdot 42374 \\
\text{pro. part} = & \overline{6}, \\
\log 2\cdot 654 = & \cdot 42390 \\
\log 2\cdot 653 = & \cdot 42374 \\
\text{pro. part} = & \overline{16},
\end{array}$$
$\frac{6}{16} = \cdot 375$,
$\therefore\ (1\cdot 05)^{20} = 2\cdot 653375$.

7. By inspection of given logs,

diff. for 1 = 330,

and ·2 of 330 = 66,

$\therefore \log 131072 = 5{\cdot}1175100$,

$\therefore \log (131072)^{\frac{1}{17}} = \frac{1}{17}(5{\cdot}11751)$

$= {\cdot}30103 = 1 - {\cdot}69897$

$= 1 - \log 5 = \log \frac{10}{5}$

$\therefore (131072)^{\frac{1}{17}} = 2$.

8. Let x = value reqd.; then

$$x = \left(\frac{2^2 . 3 . 13}{10}\right)^{\frac{1}{3}} \times \left(\frac{3^2}{2 . 10^3}\right)^{\frac{3}{4}} \times \left(\frac{10^4 . 2}{13}\right)^{\frac{1}{2}} = \frac{2^{\frac{5}{12}} . 3^{\frac{11}{6}}}{10^{\frac{7}{12}} . 13^{\frac{1}{6}}};$$

$\therefore \log x = \frac{5}{12}\log 2 + \frac{11}{6}\log 3 - \frac{7}{12} - \frac{1}{6}\log 13$

$= {\cdot}1254292 + {\cdot}8747224 - {\cdot}5833333 - {\cdot}1856572 = {\cdot}2311611;$

$\therefore$ pro. part = 230, and diff. = 255, and $\frac{230}{255} = {\cdot}9$; $\therefore x = 1{\cdot}70279$.

9. $\log c = \log(a+b) + \log\sin\frac{1}{2}C + \log\sec\phi$

$= \log(a+b) + L\sin\frac{1}{2}C - 10 + L\sec\phi - 10 = l$.

10. Squaring both sides, $2(L\tan\frac{1}{2}A - 10) = \log(s-b) + \log(s-c) - \log s - \log(s-a)$, $\therefore f = l$.

11. $\log c = \log a + \log\sin C - \log\sin A = \log a + L\sin C - 10 - (L\sin A - 10) = l$. $\log\sin C = \log c + \log\sin A - \log a$, or $L\sin C - 10 = \log c + L\sin A - 10 - \log a$, $\therefore f = l$.

12. $\log x = \log a + \log\sin\alpha + \log\tan\beta - \log\tan(\alpha+\beta)$

$= \log a + L\sin\alpha - 10 + L\tan\beta - 10 - [L\tan(\alpha+\beta) - 10] = l$.

13. diff. for 1′ = 2455,

$\frac{36}{60}$ of 2455 = 1473

cos 57° 32′ = ·5368089

cos 57° 32′ 36″ = ·5366616.

14. pro. part = 97,

$\frac{97}{1883}$ of 60″ = 3″,

angle reqd. = 47° 4′ 3″.

15. pro. part = 2314,

$\frac{2314}{3525}$ of 60″ = 40″,

angle reqd. = 75° 56′ 49″.

16. diff. for 1′ = 2226,

$\frac{36}{60}$ of 2226 = 1336

L cos 60° 25′ = 9·6934534

L cos 60° 25′ 36″ = 9·6933198.

17. $\frac{12}{60}$ of 6208 = 1242

L tan 12° = 9·3274745

L tan 12° 0′ 12″ = 9·3275987.

18. $\frac{2}{10}$ of 441 = 88

L cot 53° 32′ 40″ = 9·8685041

L cot 53° 32′ 42″ = 9·8684953.

19. pro. part = 317,

$\frac{317}{936}$ of 60″ = 20″·3,

angle reqd. = 53° 25′ 20″·3.

20. pro. part = 595,

$\frac{595}{2612}$ of 60″ = 14″,

angle reqd. = 37° 40′ 14″.

EXERCISE IX.

1. $f = 4s - 2(a+b+c) = 4s - 2 \times 2s = 0$.

2. $f = 2c\sqrt{\left(\frac{s(s-a)}{bc} \times \frac{s(s-b)}{ac} \times \frac{ab}{(s-a)(s-b)}\right)} = 2c \cdot \frac{s}{c} = 2s = l$.

3. $f = \sqrt{\left(\frac{s(s-a)}{(s-b)(s-c)} \times \frac{s(s-b)}{(s-a)(s-c)}\right)} = \frac{s}{s-c} = \frac{2s}{2(s-c)} = l$.

4. $f = 1 - \sqrt{\left(\frac{(s-b)(s-c)}{s(s-a)} \times \frac{(s-a)(s-c)}{s(s-b)}\right)} = 1 - \frac{s-c}{s} = 1 - \frac{2(s-c)}{2s}$
$= \frac{2s-2s+2c}{2s} = l.$

5. $f = \frac{a(1-\cos B)}{b(1-\cos A)} = \frac{a}{b} \cdot \frac{2\sin^2\frac{1}{2}B}{2\sin^2\frac{1}{2}A} = \frac{a}{b} \cdot \frac{(s-a)(s-c)}{ac} \cdot \frac{bc}{(s-b)(s-c)}$
$= \frac{2(s-a)}{2(s-b)} = l.$

6. $f = c\frac{(s-a)(s-c)}{ac} + b\frac{(s-a)(s-b)}{ab} = \frac{(s-a)(s-c)+(s-a)(s-b)}{a}$
$= \frac{(s-a)(s-c+s-b)}{a} = \frac{(s-a)(2s-b-c)}{a} = \frac{(s-a)\,a}{a} = l.$

7. $\therefore \tan\frac{A}{2} \pm \tan\frac{B}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}} \pm \sqrt{\left\{\frac{(s-a)(s-c)}{s(s-b)}\right\}}$
$= \frac{(s-b)\sqrt{(s-c)} \pm (s-a)\sqrt{(s-c)}}{\sqrt{[s(s-a)(s-b)]}},$
$\therefore f = \frac{\sqrt{[s-c]}(s-b+s-a)}{\sqrt{s}(s-a)(s-b)]} \times \frac{\sqrt{[s(s-a)(s-b)]}}{\sqrt{[s-c]}(s-b-s+a)} = \frac{2s-a-b}{a-b} = l.$

8. $\because \cot\frac{B}{2} + \cot\frac{C}{2} = \sqrt{\left\{\frac{s(s-b)}{(s-a)(s-c)}\right\}} + \sqrt{\left\{\frac{s(s-c)}{(s-a)(s-b)}\right\}}$
$= \frac{(s-b)\sqrt{s}+(s-c)\sqrt{s}}{\sqrt{[(s-a)(s-b)(s-c)]}} = \frac{(s-b+s-c)\sqrt{s}}{\sqrt{[(s-a)(s-b)(s-c)]}},$
$\therefore f = \sqrt{\left\{\frac{s(s-a)}{(s-b)(s-c)}\right\}} \times . \frac{\sqrt{[(s-a)(s-b)(s-c)]}}{a\sqrt{s}} = l.$

9. $s = \frac{1}{2}(4+7+9) = 10, \therefore s-b = 3; \therefore \cos\frac{B}{2} = \sqrt{\frac{10.3}{4.9}} = \frac{1}{6}\sqrt{30}.$

10. $s = \frac{1}{2}(\sqrt{3}+1+\sqrt{6}+2) = \frac{1}{2}(3+\sqrt{3}+\sqrt{6});$
$\therefore s-a = \frac{1}{2}(1-\sqrt{3}+\sqrt{6});$
$\therefore s-b = \frac{1}{2}(3+\sqrt{3}-\sqrt{6}); \quad s-c = \frac{1}{2}(-1+\sqrt{3}+\sqrt{6});$
$\therefore \cot\frac{C}{2} = \sqrt{\left\{\frac{(3+\sqrt{3}+\sqrt{6})(-1+\sqrt{3}+\sqrt{6})}{(1-\sqrt{3}+\sqrt{6})(3+\sqrt{3}-\sqrt{6})}\right\}}$
$= \sqrt{\left\{\frac{6\sqrt{2}+2\sqrt{6}+(6+2\sqrt{3})}{6\sqrt{2}+2\sqrt{6}-(6+2\sqrt{3})}\right\}}$
$= \sqrt{\left\{\frac{(6+2\sqrt{3})(\sqrt{2}+1)}{(6+2\sqrt{3})(\sqrt{2}-1)}\right\}} = \sqrt{(\sqrt{2}+1)^2} = \sqrt{2}+1.$

CHAPTER XVI.

The difference and proportional parts, except in (1), are taken by inspection, when required.

1. **Let c, A, B be the given side and angles. Then b is required. $C = 180° - (A+B) = 180° - 128° 29' 20'' = 51° 30' 40''$, and $\log b = \log c + L \sin B - L \sin C$.**

$L \sin 104° = L \sin 76°.$

$L \sin 51° 30' = 9{\cdot}8935444$

$\frac{40}{60}$ of $1004 = 669$

$L \sin 51° 30' 40'' = 9{\cdot}8936113.$

$\log c = 3{\cdot}$

$L \sin B = 9{\cdot}9869041$

$12{\cdot}9869041$

$L \sin C = 9{\cdot}8936113$

$\log b = 3{\cdot}0932928.$

$\log b = 3{\cdot}0932928$

$\log 12396 = 4{\cdot}0932816$

$112,$

$\frac{112}{350} = {\cdot}32,$

$\therefore$ side $b = 1239{\cdot}632$ ft.

2. $L \sin B = \log b - \log a + L \sin A.$

$\log b = 2{\cdot}8649855$

$L \sin A = 9{\cdot}9785334$

$12{\cdot}8435189$

$\log a = 2{\cdot}9501701$

$9{\cdot}8933488$

$L \sin 51° 28' = 9{\cdot}8933433$

$55,$

$\frac{55}{1067}$ of $60'' = 3'',$

$B = 51° 28' 3''.$

3. Suppose a, b, C given. Then

$\tan \frac{1}{2}(A-B) = \frac{a-b}{a+b} \cot \frac{1}{2}C$

$= \frac{40}{160} \cot \frac{1}{2}C = \frac{1}{2^2} \cot \frac{1}{2}C,$

$\therefore L \tan \frac{1}{2}(A-B)$

$= L \cot \frac{1}{2}C - 2 \log 2,$

$L \cot \frac{1}{2}C = 10{\cdot}4101858$

$2 \log 2 = {\cdot}60206$

$L \tan \frac{1}{2}(A-B) = 9{\cdot}8081258$

pro. part $= 429,$

diff. for $1' = 2777,$

$\frac{429}{2777}$ of $60'' = 9'',$

$\frac{1}{2}(A-B) = 32° 44' 9''$

and $\frac{1}{2}(A+B) = 68° 45'$

$\therefore A = 101° 29' 9''$

$B = 36° 0' 51''.$

4. $s = \frac{1}{2}(18+20+22) = 30,$

$s-a = 12,\ s-b = 10,\ s-c = 8.$

$\therefore \tan^2 \frac{1}{2}A = \frac{10 \,.\, 8}{30 \,.\, 12} = \frac{2}{3^2},$

$\therefore 2L \tan \frac{1}{2}A = \log 2 - 2\log 3 + 20$

$= 20{\cdot}30103 - {\cdot}9542426$

$= 19{\cdot}3467874,$

$\therefore L \tan \frac{1}{2}A = 9{\cdot}6733937.$

5. $\tan \frac{1}{2}(A-B) = \frac{a-b}{a+b} \cot \frac{1}{2}C = \frac{16}{20} \cot \frac{1}{2}C = \frac{2^3}{10} \cot \frac{1}{2}C;$

$\therefore L \tan \frac{1}{2}(A-B) = 3 \log 2 + L \cot 27° 30' - 1 = {\cdot}90309 + 10{\cdot}2835233 - 1 = 10{\cdot}1866133;$ $\therefore \frac{1}{2}(A-B) = 56° 56' + \frac{2364}{2763}$ of $60'' = 56° 56' 51''{\cdot}3$, and $\frac{1}{2}(A+B) = 90° - 27° 30' = 62° 30';$

$\therefore A = 119° 26' 51''{\cdot}3, \quad B = 5° 33' 8''{\cdot}7.$

6. $s = \frac{1}{2}(4+5+6) = \frac{15}{2}, \quad s-b = \frac{5}{2}, \quad \cos^2 \frac{1}{2}B = (\frac{15}{2} \,.\, \frac{5}{2}) \div (4 \,.\, 6) = \frac{25}{32} = \frac{100}{2^7};$ $\therefore 2L \cos \frac{1}{2}B = 2 - 7 \log 2 + 20 = 22 - 2{\cdot}10721 = 19{\cdot}89279,$

$\therefore L \cos \frac{1}{2}B = 9{\cdot}9463950 = L \cos 27° 53' - 90 = L \cos 27° 53' 8''$, for $\frac{90}{669}$ of $60'' = 8''$; $\therefore B = 55° 46' 16''.$

7. $\tan \frac{1}{2}(A-B) = \frac{9-7}{9+7} \cot 32° 6' = \frac{1}{2^3} \cot 32° 6';$ $\therefore L \tan \frac{1}{2}(A-B) = L \cot 32° 6' - 3 \log 2 = 10{\cdot}2025255 - {\cdot}90309 = 9{\cdot}2994355 = L \tan 11° 16' 10''$, for pro. part $= 1139$, and $\frac{1139}{6858}$ of $60'' = 10''$;

$\therefore A = \frac{1}{2}(A+B) - \frac{1}{2}(A-B) = 69° 10' 10'',$

$B = \frac{1}{2}(A+B) - \frac{1}{2}(A-B) = 46° 37' 50''.$

8. $\tan\frac{1}{2}(A-B) = \frac{9-7}{9+7}\cot\frac{1}{2}C = \frac{1}{2^3}\cot 23^\circ 42' 30''$, $\therefore$ $L\tan\frac{1}{2}(A-B)$
$= L\tan 66^\circ 17' 30'' - 3\log 2 = 10{\cdot}3573942 - {\cdot}90309 = 9{\cdot}4543042$
$= L\tan 15^\circ 53' 20''$, for pro. part $= 1563$, and $\frac{1563}{4797}$ of $60'' = 20''$,
$\therefore$ $\frac{1}{2}(A-B) = 15^\circ 53' 20''$, and $\frac{1}{2}(A+B) = 66^\circ 17' 30''$;
$\therefore$ $A = 82^\circ 10' 50''$, $B = 50^\circ 24' 10''$.

9. $s = 12$, $s-a = 5$, &c., $\therefore$ $\tan^2\frac{1}{2}A = \frac{(s-b)(s-c)}{s(s-a)} = \frac{4\,.\,3}{12\,.\,5} = \frac{2}{10}$;
$\therefore$ $L\tan\frac{1}{2}A = \frac{1}{2}(\log 2 - 1 + 20) = \frac{1}{2}(19{\cdot}30103) = 9{\cdot}650515$,
$\therefore$ pro. part $= 81$, and $\frac{81}{565}$ of $10'' = 1''{\cdot}5$ nearly;
$\therefore$ $A = (24^\circ 5' 41''{\cdot}5) \times 2 = 48^\circ 11' 23''$.

Also, $\tan^2\frac{1}{2}B = \frac{5\,.\,3}{12\,.\,4} = \frac{10}{32}$; $\therefore$ $L\tan\frac{1}{2}B = \frac{1}{2}(1 - 5\log 2 + 20)$
$= 9{\cdot}747425$, $\therefore$ pro. part $= 67$, and $\frac{67}{494}$ of $10'' = 1''{\cdot}5$ nearly;
$\therefore$ $B = (29^\circ 12' 21''{\cdot}5) \times 2 = 58^\circ 24' 43''$,
$\therefore$ $C = 180^\circ - (A+B) = 73^\circ 23' 54''$.

10. Suppose $B = 60^\circ$, $\therefore$ $\tan\frac{1}{2}(A-C) = \frac{19-1}{19+1}\cot 30^\circ = \frac{9\sqrt{3}}{10} = \frac{3^{\frac{5}{2}}}{10}$,
$\therefore$ $L\tan\frac{1}{2}(A-C) = \frac{5}{2}\log 3 - 1 + 10 = 10{\cdot}1928032$, $\therefore$ $\frac{1}{2}(A-C) =$
$57^\circ 19' 11''$, and $\frac{1}{2}(A+C) = 60^\circ$, $\therefore$ $A = 117^\circ 19' 11''$, $B = 2^\circ 40' 49''$.

11. $L\sin B = \log b + L\sin A - \log a = 1{\cdot}8770256 + 9{\cdot}7735727 - 1{\cdot}8080759$
$= 9{\cdot}8425224$; and $B = 44^\circ 5' 45''$; and as $b > a$, B also may $=$
$135^\circ 54' 15''$.

12. $\log b = \log a + L\sin B - L\sin(B+C) = \log a + L\sin 16^\circ - L\sin 80^\circ$
$= 2{\cdot}4048337 + 9{\cdot}4403381 - 9{\cdot}9933515 = 1{\cdot}8518203$,
$\therefore$ pro. part $= 118$, and diff. $= 611$, and $\frac{118}{611} = {\cdot}193$; $\therefore$ $b = 71{\cdot}09193$.

13. $\because a^2 = c^2 - b^2 = (c+b)(c-b)$, $\therefore$ $2\log a = \log(c+b) + \log(c-b)$, $\therefore$ &c.

14. (i.) Obtain A (or B); thus $\tan A = \frac{a}{b}$, $\therefore$ $L\tan A = \log a - \log b + 10$.

(ii.) $B = 90^\circ - A$. (iii.) $c = a\operatorname{cosec} A$, $\therefore$ $\log c = \log a + L\operatorname{cosec} A - 10$.

15. $\tan A = \frac{9{\cdot}65}{12{\cdot}24} = \frac{965}{1224} = \frac{193 \times 5}{153 \times 8} = \frac{193 \times 10}{153 \times 2^4}$,
$\therefore$ $L\tan A = 2{\cdot}2855573 + 1 - 2{\cdot}1846914 - 1{\cdot}20412 + 10 = 9{\cdot}8967459$;
$\therefore$ pro. part $= 343$, and diff. $= 2598$; also $\frac{343}{2598}$ of $60'' = 8''$;
$\therefore$ $A = 38^\circ 15' 8''$, and $B = 51^\circ 44' 52''$.

CHAPTER XVII.

1. Suppose A, B, fig. (13), the cottages, CD the hill; make DE horizontal. Given $AB = 176$ yd., $EDA = 30^\circ$, $EDB = 45^\circ$. Hence $DAC = 30^\circ$, $DBC = 45^\circ$, $ADB = 45^\circ - 30^\circ = 15^\circ$, $\therefore$ $CD = DB\sin 45^\circ$
$= AB\frac{\sin 30^\circ}{\sin 15^\circ}\frac{1}{\sqrt{2}} = 176 \text{ yd.} \times \frac{1}{2} \times \frac{2\sqrt{2}}{\sqrt{3}-1} \times \frac{1}{\sqrt{2}} = 88(\sqrt{3}+1)$ yd.

2. In fig. (5), let $CAB = BAD = A$, $BD = a$ ft. Then $BC = BA\sin A$
$= BD\frac{\sin D \sin A}{\sin DAB} = a\cos CAD = a\cos 2A$.

3. Suppose (13) the fig.; then $AB = BD\dfrac{\sin 30^\circ}{\sin 15^\circ} = \dfrac{CD}{\sin 45^\circ} \cdot \dfrac{\sin 30^\circ}{\sin 15^\circ}$
$= 50 \text{ ft.} \times \sqrt{2} \times \frac{1}{2} \times \dfrac{2\sqrt{2}}{\sqrt{3}-1} = 50(\sqrt{3}+1)$ ft.

4. In fig. (15), $PQH = QHE = 3a$, $QRH = RHE = a$, $QHR = 2a$, $QR = a$;
$\therefore\ HP = HQ \sin 3a = QR\dfrac{\sin a \sin 3a}{\sin 2a} = a\dfrac{\sin 3a}{2\cos a}$.

5. In fig. (15), suppose Q, R the two positions of ship; then $QR = 18$ m.;
$\therefore\ QH = QR\dfrac{\sin 45^\circ}{\sin 30^\circ} = 18 \text{ m.} \times \dfrac{1}{\sqrt{2}} \cdot \dfrac{2}{1} = 18\sqrt{2}$ m.

6. In fig. (16), $Ss = 14$ m., $LSs = 105^\circ$, $SsL = 45^\circ$, $\therefore\ SLs = 30^\circ$;
$\therefore\ SL = Ss\dfrac{\sin 45^\circ}{\sin 30^\circ} = 14 \text{ m.} \dfrac{1}{\sqrt{2}} \cdot \dfrac{2}{1} = 14\sqrt{2}$ m.
$Ls = 14 \text{ m.} \dfrac{\sqrt{3}+1}{2\sqrt{2}} \cdot \dfrac{2}{1} = 7\sqrt{2}(\sqrt{3}+1)$ m.

7. In fig. (17), let $PB = 12$ m., $PN = 8$ m, S be the point of meeting; then $AP = BP \operatorname{cosec} 45^\circ = 12\sqrt{2}$ m.; $\therefore\ AN = (12\sqrt{2} - 8)$ m.;
$\therefore\ NS = AN\dfrac{\sin 135^\circ}{\sin 30^\circ} = 4\sqrt{2}(3-\sqrt{2}) \text{ m.} \dfrac{1}{\sqrt{2}} \cdot \dfrac{2}{1} = 8(3-\sqrt{2})$ m.
$AS = 4\sqrt{2}(3-\sqrt{2}) \text{ m.} \dfrac{\sqrt{3}-1}{2\sqrt{2}} \cdot \dfrac{2}{1} = 4(3\sqrt{3} - \sqrt{6} - 3 + \sqrt{2})$ m.;
$\therefore\ BS = 12 + AS = 4(3\sqrt{3} - \sqrt{6} + \sqrt{2})$ m., $PN + NS = 8(4 - \sqrt{2})$ m.

8. Suppose (18) the fig., then $AD = AB \sin 60^\circ = BC\dfrac{\sin 135^\circ}{\sin 15^\circ} \cdot \dfrac{\sqrt{3}}{2}$
$= 1 \text{ m.} \dfrac{1}{\sqrt{2}} \cdot \dfrac{2\sqrt{2}}{\sqrt{3}-1} \cdot \dfrac{\sqrt{3}}{2} = \dfrac{1}{2}(3+\sqrt{3})$ m.

9. In fig. (19), let AB, CD be the towers, and $ADB = 45^\circ$; then $AE = AB = x$ suppose, $\therefore\ AEB = 45^\circ$, $\therefore$ a circle will pass through B, A, E, D; in which $BDE = 90^\circ$ for $BAE = 90^\circ$. Hence $BE^2 = BD^2 + DE^2$ or $(x\sqrt{2})^2 = (x-a)^2 + (x+c)^2 + a^2 + c^2$, for $BF = x - a$, and $DF = x + c$; $\therefore\ 2ax - 2cx = 2(a^2 + c^2)$, $\therefore\ x = \dfrac{a^2+c^2}{a-c}$.

10. Suppose (20) the fig.; then if $CD = x$, $AC = x \cot 30^\circ = x\sqrt{3}$, and $BC = x \cot 18^\circ$. But $BAC = 90^\circ$, $\therefore\ (x\sqrt{3})^2 + a^2 = x^2 \cot^2 18^\circ$
$= x^2\dfrac{5+\sqrt{5}}{8} \times \dfrac{16}{6-2\sqrt{5}} = x^2\dfrac{(5+\sqrt{5})(6+2\sqrt{5})}{8} = x^2(5+2\sqrt{5})$,
$\therefore\ x^2(2+2\sqrt{5}) = a^2$, $\therefore\ x =$ &c.

11. Suppose (21) the fig.; then $DH = DB = x$ suppose. Now $\tan A = \cot DCH$, $\therefore\ \dfrac{x}{x+a} = \dfrac{x-b}{x}$, $\therefore\ x^2 = x^2 + (a-b)x - ab$, $\therefore\ x = \dfrac{ab}{a-b}$.

12. Suppose (21) the fig.; then $HCD = 2A$, $HBC = 90^\circ - A$,
$AHB = 90^\circ - 2A$; $\therefore\ CD = CH \cos 2A = BH\dfrac{\sin(90^\circ - A)}{\sin 2A}\cos 2A$
$= AB\dfrac{\sin A \cos A \cos 2A}{\sin(90^\circ - 2A)\sin 2A} = \dfrac{a}{2}$.

13. Take fig. to Art. 136 (3); then $AB = 2$ y., $OC = 11$ y., $\tan AOB = \frac{1}{10}$;
$\therefore \dfrac{\frac{2}{11}}{1 + \frac{1}{121}(x^2 + 2x)} = \dfrac{1}{10}$; $\therefore x^2 + 2x - 99 = 0$ or $(x+11)(x-9) = 0$,
$\therefore x = 9$.

14. Suppose (22) the fig., and let $\tan DAB = x$, $\therefore x = \tan(BAC + CAD)$
$= \dfrac{\frac{1}{2}x + \frac{1}{3}}{1 - \frac{1}{6}x} = \dfrac{3x + 2}{6 - x}$, $\therefore x^2 - 3x + 2 = 0$ or $(x-1)(x-2) = 0$,
$\therefore A = 1$ or 2.

15. With fig. to Art. 136 (3), let $x =$ breadth reqd.; then $POC = AOB$
$= AOC - BOC$. $\dfrac{6}{x} = \left(\dfrac{230}{x} - \dfrac{200}{x}\right) \div \left(1 + \dfrac{230 \cdot 200}{x^2}\right) =$
$\dfrac{30}{x} \times \dfrac{x^2}{x^2 + 46000} = \dfrac{30x}{x^2 + 46000}$, $\therefore 24x^2 = 276000$, $\therefore x^2 = 11500$,
$\therefore x = 10\sqrt{115}$.

16. Suppose (23) the fig.; then $AB = AD\dfrac{\sin BDA}{\sin ABD}$
$= DE\dfrac{\sin E}{\sin EAD} \cdot \dfrac{\sin(\beta - \alpha)}{\sin\beta} = a\dfrac{\cos\theta}{\sin(\theta - \alpha)} \cdot \dfrac{\sin(\beta - \alpha)}{\sin\beta}$.

17. Suppose (24) the fig., and let $BC = x$, then $AC = x\cot\alpha$, $DC = x\cot\beta$,
and $\because CAD = 90°$, $\therefore a^2 = x^2\cot^2\beta - x^2\cot^2\alpha = x^2(\operatorname{cosec}^2\beta - \operatorname{cosec}^2\alpha)$
$= x^2\dfrac{\sin^2\alpha - \sin^2\beta}{\sin^2\alpha\sin^2\beta} = x^2\dfrac{\sin(\alpha + \beta)\sin(\alpha - \beta)}{\sin^2\alpha\sin^2\beta}$,
$\therefore x = \dfrac{a\sin\alpha\sin\beta}{\sqrt{\sin(\alpha + \beta)\sin(\alpha - \beta)}}$.

18. Construct as in (25); and let $CH = x$, $BP = y$. By Geometry, since $CAH = CBH$, a circle will pass round $ABCH$, and therefore
$y(y + 20) = 20(x + 20)$ ....(1).
Also $1 = \tan CBH = \left(\dfrac{x + 20}{y} - \dfrac{20}{y}\right) \div \left(1 + \dfrac{(x + 20)\,20}{y^2}\right)$
$= \dfrac{xy}{y^2 + (x + 20)\,20} = \dfrac{xy}{y^2 + y(y + 20)}$ [by (1)] $= \dfrac{x}{2y + 20}$;
$\therefore 2y + 20 = x$, $\therefore 2(y + 20) = x + 20$ ....(2).
From (1) ÷ (2), $y = 40$, $\therefore x = 100$.

19. Construct as in (26), and let $CD = x$, $BC = AD = y$;
$\therefore x = DE - CE = y(\sin A - \sin B) = y \cdot 2\sin\frac{1}{2}(A - B)\cos\frac{1}{2}(A + B)$...(1),
and $a = BE - AE = y(\cos B - \cos A) = y \cdot 2\sin\frac{1}{2}(A - B)\sin\frac{1}{2}(A + B)$...(2).
Dividing (1) by (2), $x = a\cot\frac{1}{2}(A + B)$.

20. Given, in triangle ABC, $c = 100$ yds., $A = 53°\,20'$, $B = 59°\,30'$; reqd. b. $C = 180° - (A + B) = 67°\,10'$, $\sin C = \cos 22°\,50'$; $\log b = \log c + L\sin b - L\sin C = 2 + 9{\cdot}9353204 - 9{\cdot}9645602 = 1{\cdot}9707602$;
$\therefore b = 93{\cdot}489$ yds.

21. Standing at the extremity of the shadow, the sun will be seen in a line with the top of the tower. Its angular altitude is therefore that of the top of the tower. Suppose it $= \alpha$. Then $\tan\alpha = \frac{150}{75} = 2$, $\therefore L\tan\alpha = \log 2 + 10 = 10{\cdot}3010300$. Hence pro. part $= 306$, diff. for $60'' = 3159$, $\frac{306}{3159}$ of $60'' = 6''$, $\therefore \alpha = 63°\,26'\,6''$.

22. Given (fig. 11) $DA = 30$ ft., $BAC = 51°$, $BDC = 46°$; reqd. AC, $= x$ feet, suppose.

$ABD = BAC - D = 5°.$

$$x = AB \cos 51° = AD \frac{\sin 46°}{\sin 5°} \cos 51°,$$

and $\cos 51° = \sin 39°$.

$\log 30 = \log 3 + \log 10 = 1{\cdot}47712.$

$x = 155{\cdot}8.$

$$\begin{aligned} \log x &= \log 30 + L \sin 46° + L \cos 51° \\ &\qquad - L \sin 5° - 10 \\ &= 1{\cdot}47712 \\ &\quad + 9{\cdot}85693 \\ &\quad + 9{\cdot}79887 - 18{\cdot}94029 \\ &= 21{\cdot}13292 - 18{\cdot}94029 \\ &= 2{\cdot}19263. \end{aligned}$$

23. Given (fig. 11) $DA = 600$ yds. $D = 30°$, $BAC = 60°$; reqd. BC.

$ABD = BAC - D = 30° = D,$

$\therefore AB = AD,$

$$\begin{aligned} \therefore BC &= AB \sin 60° \\ &= 600 \times \tfrac{1}{2}\sqrt{3} = 300\sqrt{3}. \end{aligned}$$

$$\begin{aligned} \log BC &= \log 3 + \log 100 + \tfrac{1}{2} \log 3 \\ &= 2{\cdot}47712 \\ &\quad + {\cdot}23856 \\ &= 2{\cdot}71568. \end{aligned}$$

pro. pt. $= 811$, diff. $= 843$, $\frac{811}{843} = {\cdot}96$,

$BC = 519{\cdot}6.$

24. Given a, b, C; reqd. A, B.

$$\tan \tfrac{1}{2}(A - B) = \frac{a-b}{a+b} \cot \tfrac{1}{2}C$$

$$= \frac{21 - 10\frac{1}{2}}{21 + 10\frac{1}{2}} \cot \tfrac{1}{2}C = \tfrac{1}{3} \cot \tfrac{1}{2}C.$$

$\log 3 = \log \frac{30}{10}$

$= \log 15 + \log 2 - \log 10 = {\cdot}4771213.$

$$\begin{aligned} L \tan \tfrac{1}{2}(A - B) &= L \cot \tfrac{1}{2}C - \log 3 \\ &= 10{\cdot}4771213 \\ &\quad - {\cdot}4771213 \\ &= 10, \end{aligned}$$

$\therefore \log \tan \tfrac{1}{2}(A - B) = 0 = \log 1.$

$\therefore \tfrac{1}{2}(A + B) = 90° - \tfrac{1}{2}C = 71° 33' 54''$, $\tfrac{1}{2}(A - B) = 45°$; $\therefore$ adding and subtracting, $A = 116° 33' 54''$, $B = 26° 33' 54''$.

25. Given (fig. 27) $AP = 220$ ft., $BAC = 71°$, $BPC = 55°$; reqd. BC, BA.

$$\begin{aligned} BC &= AC \sin CAB \\ &= AP \frac{\sin CPA}{\sin ACP} \sin CAB \end{aligned}$$

$$\begin{aligned} \log BC &= \log AP + L \sin 55° \\ &\quad + L \sin 71° - L \sin 16° - 10 \\ &= 2{\cdot}34242 \\ &\quad + 9{\cdot}91336 \\ &\quad + 9{\cdot}97567 - 19{\cdot}44034 \\ &= 22{\cdot}23145 - 19{\cdot}44034 \\ &= 2{\cdot}79111; \end{aligned}$$

$\therefore BC = 618{\cdot}17$ ft.

$$\begin{aligned} BA &= AC \cos CAB \\ &= AP \frac{\sin CPA}{\sin ACB} \cos CAB \end{aligned}$$

$$\begin{aligned} \log BA &= \log AP + L \sin 55° \\ &\quad + L \cos 71° - L \sin 16° - 10 \\ &= 2{\cdot}34242 \\ &\quad + 9{\cdot}91336 \\ &\quad + 9{\cdot}51264 - 19{\cdot}44034 \\ &= 21{\cdot}76842 - 19{\cdot}44034 \\ &= 2{\cdot}32808; \end{aligned}$$

$\therefore BA = 212{\cdot}85$ ft.

26. Given (fig. 28) $BD = 180$ yds., $BDP = 41°$, $BDQ = 68°$.

$DP = BD \sec BDP,$

$$\begin{aligned} \log DP &= \log 180 - L \cos 41° + 10 \\ &= 12{\cdot}255273 \\ &\quad - 9{\cdot}877780 \\ &= 2{\cdot}377493; \end{aligned}$$

$\therefore BP = 238{\cdot}5$ yds.

$DQ = BD \sec BDQ,$

$$\begin{aligned} \log DQ &= \log 180 - L \cos 68° + 10 \\ &= 12{\cdot}255273 \\ &\quad - 9{\cdot}573575 \\ &= 2{\cdot}681698; \end{aligned}$$

$\therefore BQ = 480{\cdot}5$ yds.

27. Given (fig. 29) $CD = 64$ ft., $BDC = 80° = BCD$, $ACB = 30°$, $BAC = 20°$; reqd. AB. $AB = BC \dfrac{\sin 30°}{\sin 20°} = CD \dfrac{\sin 80°}{\sin 20°} \dfrac{\sin 30°}{\sin 20°}$

$$= 64 \frac{4 \cos 40° . \sin 20° \cos 20°}{\sin 20° \sin 20°} \frac{1}{2} = 128 . \cos 40° \cot 20°,$$

$\therefore \log AB = 7 \log 2 + L \cos 40° - L \tan 20° = 2{\cdot}1072100 + 9{\cdot}8842540 - 9{\cdot}5610659 = 11{\cdot}9914640 - 9{\cdot}5610659 = 2{\cdot}4303981$,

$\therefore AB = 296{\cdot}40031$.

28. The formula $\cot \frac{1}{2}C = \dfrac{a+b}{a-b} \tan \frac{1}{2}(A-B)$ will determine $\frac{1}{2}C$; then $\frac{1}{2}(A+B) = 90° - \frac{1}{2}C$, and is known; and $A = \frac{1}{2}(A+B) + \frac{1}{2}(A-B)$, $B = \frac{1}{2}(A+B) - \frac{1}{2}(A-B)$, and are both known. Side c follows from $c = a \dfrac{\sin C}{\sin A}$.

29. $C = 180° - (A+B)$; then $a - b$ is given by $a - b = (a+b) \dfrac{\tan \frac{1}{2}(A-B)}{\cot \frac{1}{2}C}$. Sides a, b follow, and the solution is in ordinary course.

30. $\sin \frac{1}{2}(A-B) = \dfrac{a-b}{c} \cos \frac{1}{2}C$ gives $\frac{1}{2}(A-B)$, and (Euc. I. 32) $\frac{1}{2}(A+B)$; after which a and b follow, and then a, b, in ordinary way.

31. $\frac{1}{2}(b+c-a) = s-a$, $\frac{1}{2}[b-(c-a)] = s-c$; hence $\tan \frac{1}{2}C = \dfrac{s-a}{s-c} \tan \frac{1}{2}A$ will give C, and the remaining solution follows.

32. $\frac{1}{2}(b+c+a) = s$, $\frac{1}{2}(c+a-b) = s-b$; then $\cot \frac{1}{2}C = \dfrac{s}{s-b} \tan \frac{1}{2}A$ gives C, and the rest follow as usual.

CHAPTER XVIII.

1. $f = \sqrt{\left(\dfrac{\Delta}{s} . \dfrac{\Delta}{s-a} . \dfrac{\Delta}{s-b} . \dfrac{\Delta}{s-c}\right)} = \dfrac{\Delta^2}{\Delta} = l.$

2. $l = 4 \dfrac{abc}{4\Delta} . \dfrac{\Delta}{s} . s = f.$ **3.** $f = \dfrac{bc}{2s} . \dfrac{2\Delta}{bc} = \dfrac{\Delta}{s} = l.$

4. $f = \dfrac{2\Delta}{bc} . \dfrac{bc}{2(s-a)} = \dfrac{\Delta}{s-a} = l.$

5. $l = \sqrt{\left\{\dfrac{(s-b)(s-c)}{bc}\right\}} \times \sqrt{[bc . s(s-a)]} = \sqrt{[s(s-a)(s-b)(s-c)]} = f.$

6. $l = \dfrac{\Delta}{s} . \dfrac{\Delta}{s-a} . \dfrac{s-b}{\Delta} . \dfrac{s-c}{\Delta} = \dfrac{(s-b)(s-c)}{s(s-a)} = f.$

7. $a \sin \frac{1}{2}B \sin \frac{1}{2}C \sec \frac{1}{2}A = a \sqrt{\left\{\dfrac{(s-a)(s-c)}{ac} . \dfrac{(s-a)(s-b)}{ab} . \dfrac{bc}{s(s-a)}\right\}}$

$$= a \sqrt{\left\{\frac{(s-a)(s-b)(s-c)}{a^2 . s}\right\}} = \frac{\Delta}{s} = r.$$

$$b \sin \tfrac{1}{2}C \sin \tfrac{1}{2}A \sec \tfrac{1}{2}B = b \sqrt{\left\{\frac{(s-a)(s-b)}{ab} \cdot \frac{(s-b)(s-c)}{bc} \cdot \frac{ac}{s(s-b)}\right\}}$$
$$= b \times \frac{1}{b} \sqrt{\left\{\frac{(s-a)(s-b)(s-c)}{s}\right\}} = r.$$

$$c \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sec \tfrac{1}{2}C = c \sqrt{\left\{\frac{(s-b)(s-c)}{bc} \cdot \frac{(s-a)(s-c)}{ac} \cdot \frac{ab}{s(s-c)}\right\}}$$
$$= c \times \frac{1}{c} \sqrt{\left\{\frac{(s-a)(s-b)(s-c)}{s}\right\}} = r.$$

8. $$a \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C \sec \tfrac{1}{2}A = a \sqrt{\left\{\frac{s(s-b)}{ac} \cdot \frac{s(s-c)}{ab} \cdot \frac{bc}{s(s-a)}\right\}}$$
$$= a \times \frac{1}{a} \sqrt{\left\{\frac{s(s-b)(s-c)}{s-a}\right\}} = r_a.$$

9. $$l = s \cdot c \div \left(\frac{\cos \frac{1}{2}A}{\sin \frac{1}{2}A} + \frac{\cos \frac{1}{2}B}{\sin \frac{1}{2}B}\right) = s \cdot c \times \frac{\sin \frac{1}{2}A \sin \frac{1}{2}B}{\cos \frac{1}{2}A \sin \frac{1}{2}B + \cos \frac{1}{2}B \sin \frac{1}{2}A}$$
$$= \frac{s \cdot c \cdot \sin \frac{1}{2}A \sin \frac{1}{2}B}{\sin \frac{1}{2}(A+B)} = s \cdot c \frac{\sin \frac{1}{2}A \sin \frac{1}{2}B}{\cos \frac{1}{2}C} = s \cdot r \,[\text{see } (7)] = f.$$

10. $$l = \frac{a^2}{2}\left(1 - \frac{b^2}{a^2}\right) \frac{\sin A \sin B}{\sin(A-B)} = \frac{a^2}{2}\left(1 - \frac{\sin^2 B}{\sin^2 A}\right) \frac{\sin A \sin B}{\sin(A-B)}$$
$$= \frac{a^2}{2} \frac{\sin^2 A - \sin^2 B}{\sin^2 A} \frac{\sin A \sin B}{\sin(A-B)} = \frac{a^2}{2} \frac{\sin(A+B) \sin(A-B)}{\sin^2 A}$$
$$\times \frac{\sin A \sin B}{\sin(A-B)} = \frac{a^2}{2} \frac{\sin(A+B)}{\sin A} \sin B = \frac{a^2}{2} \frac{\sin B \sin C}{\sin A} = f.$$

11. $$l = 2s \sqrt{\left\{\frac{bc}{s(s-a)} \cdot \frac{ac}{s(s-b)} \cdot \frac{ab}{s(s-c)}\right\}}$$
$$= 2s \cdot \frac{abc}{s} \sqrt{\left\{\frac{1}{s(s-a)(s-b)(s-c)}\right\}} = 8 \frac{abc}{4\Delta} = f.$$

12. $$l = s^2 \sqrt{\left\{\frac{(s-b)(s-c)(s-c)(s-a)(s-a)(s-b)}{s(s-a) \cdot s(s-b) \cdot s(s-c)}\right\}}$$
$$= \frac{s^2}{s^2} \sqrt{[s(s-a)(s-b)(s-c)]} = f.$$

13. $$l = \tfrac{1}{2}r(2R \sin A + 2R \sin B + 2R \sin C) = \tfrac{1}{2}r(a+b+c) = \frac{\Delta}{s} \cdot s = f.$$

14. $$f = 4 \frac{abc}{4\Delta} \sqrt{\left\{\frac{(s-b)(s-c)(s-c)(s-a)(s-a)(s-b)}{bc \cdot ca \cdot ab}\right\}}$$
$$= \frac{1}{\Delta} \cdot \frac{\Delta^2}{s} = \frac{\Delta}{s} = l.$$

15. $$f = 4 \frac{abc}{4\Delta} \sqrt{\left\{\frac{(s-b)(s-c) \cdot s(s-b) \cdot s(s-c)}{bc \cdot ca \cdot ab}\right\}} = \frac{1}{\Delta} \cdot \frac{\Delta^2}{s-a} = \frac{\Delta}{s-a} = l.$$

16. $$f = s(s-a) + s(s-b) + s(s-c)$$
$$= \frac{s(s-a)(s-b)(s-c)}{(s-b)(s-c)} + \frac{s(s-a)(s-b)(s-c)}{(s-c)(s-a)} + \frac{s(s-a)(s-b)(s-c)}{(s-a)(s-b)}$$
$$= \frac{\Delta}{s-b} \cdot \frac{\Delta}{s-c} + \frac{\Delta}{s-c} \cdot \frac{\Delta}{s-a} + \frac{\Delta}{s-a} \cdot \frac{\Delta}{s-b} = l.$$

17. $4\Delta\cot A = 4\,.\,\frac{1}{2}bc\sin A\cot A = 2bc\cos A = b^2+c^2-a^2$;
$\therefore\ 4\Delta(\cot A+\cot B+\cot C) = b^2+c^2-a^2+c^2+a^2-b^2+a^2+b^2-c^2$
$= a^2+b^2+c^2$; $\therefore f = l$.

18. By XII., Ex. 24, $f = 2a\sin B\sin C = 2R\sin A\times 2\sin B\sin C = l$.

19. $f = -\frac{s}{\Delta}+\frac{s-a}{\Delta}+\frac{s-b}{\Delta}+\frac{s-c}{\Delta} = \frac{-s+s-a+s-b+s-c}{\Delta}$
$= \frac{2s-(a+b+c)}{\Delta} = l$.

20. $f = \Delta\left(-\frac{1}{s}+\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\right) = \Delta\left(\frac{-s+a+s}{s(s-a)}+\frac{s-c+s-b}{(s-b)(s-c)}\right)$
$= \Delta a\left\{\frac{1}{s(s-a)}+\frac{1}{(s-b)(s-c)}\right\} = \Delta a\,\frac{s^2-(b+c)s+bc+s^2-as}{s(s-a)(s-b)(s-c)}$
$= \frac{a}{\Delta}[2s^2+bc-(b+c+a)s] = \frac{abc}{\Delta} = l$.

21. $R+r = \frac{c}{2\sin C}+\frac{2\Delta}{2s} = \frac{c}{2}+\frac{ab}{a+b+c} = \frac{ca+cb+c^2+2ab}{2(a+b+c)}$
$= \frac{c(a+b)+a^2+b^2+2ab}{2(a+b+c)} = \frac{(a+b)c+(a+b)^2}{2(a+b+c)} = \frac{(a+b)(c+a+b)}{2(a+b+c)} = \frac{a+b}{2}$.

22. $s(s-c) = \frac{(a+b+c)(a+b-c)}{2\,.\,2} = \frac{(a+b)^2-c^2}{4} = \frac{a^2+b^2+2ab-c^2}{4}$
$= \frac{1}{2}ab = \Delta$.

23. $\because\ s = \frac{1}{2}(7+8+9) = 12$, $\therefore\ \Delta = \sqrt{(12\,.\,5\,.\,4\,.\,3)} = 26{\cdot}83$.

24. $\sin B = \frac{b}{a}\sin a = \frac{\sqrt{2}}{\sqrt{3}}\,\frac{\sqrt{3}}{3} = \frac{1}{\sqrt{2}}$, $\therefore\ B = 45^\circ$ and $C = 75^\circ$;
$\therefore\ \Delta ABC = \frac{1}{2}ab\sin C = \frac{1}{2}\sqrt{3}\sqrt{2}\,\frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{1}{4}(3+\sqrt{3})$.

25. $\Delta = \frac{1}{2}\,20\,.\,20\,.\,\frac{1}{2}\sqrt{3} = 100\sqrt{3} = 100\times 1{\cdot}732 = 173{\cdot}2$.

26. Area of first triangle $= \frac{1}{2}\,.\,3\,.\,12\,.\,\frac{1}{2} = 9$. Hence, if x = hyp. reqd., by (3), p. 109, $\frac{1}{2}x^2\,.\,\frac{1}{\sqrt{2}}\,.\,\frac{1}{\sqrt{2}}\,.\,1 = 9$, $\therefore\ x^2 = 36$, $\therefore\ x = 6$.

27. By Euc. I. 34, reqd. area $= 2\,.\,\frac{1}{2}ab\sin\alpha = ab\sin\alpha$.

28. By expression just found, areas are as
$20\,.\,30\,.\,\sin 60^\circ : 30\,.\,40\,.\,\sin 120^\circ = 20\,.\,30 : 30\,.\,40 = 1:2$.

29. Let d be cut at intersection of diagonals into the parts u, v, and d' into x, y. Then, remembering (2), p. 50, area of the quadrilateral $= \frac{1}{2}(ux+xv+vy+yu)\sin\alpha = \frac{1}{2}(u+v)(x+y)\sin\alpha = \frac{1}{2}dd'\sin\alpha$.

30. Suppose CD drawn, in fig. 2, p. 73, bisecting BB' at right angles. Then, evidently, $\Delta CDB = \Delta CDB'$, and $\therefore\ \Delta CAB+\Delta CAB' = 2\Delta CAD = AD\,.\,DC = b\cos A\,.\,b\sin A$.

31. In fig. p. 110, $AO = OF\operatorname{cosec}\frac{1}{2}A = \frac{2\Delta}{2s\,.\,\sin\frac{1}{2}A} = \frac{bc\sin A}{2s\,.\,\sin\frac{1}{2}A} = l$.

32. $\because\ \frac{a}{2\sin A} = R$, $\therefore\ a = 2R\sin A$. Similarly, $b = 2R\sin B$, $c = 2R\sin C$.

CHAPTER XIX.

1. Reqd. circumf. $= 2\pi \cdot 2$ ft. $= 3{\cdot}1416 \times 4$ ft. $= 12{\cdot}5664$ ft.

2. Area of circus $= \pi \cdot 35^2$ sq. yds.; area of inner ring $= \pi \cdot 21^2$ sq. yds.; $\therefore$ area of accommodation $= \pi(35^2 - 21^2)$ sq. yds. $= \pi(35+21)(35-21)$ sq. yds. $= \frac{22}{7} \times 56 \times 14$ sq. yds. $= 2464$ sq. yds.; $\therefore$ reqd. no. $= \frac{1}{2}$ of $2464 = 1232$.

3. Let R be radius; then $1 = 2R \sin \frac{1}{5}\pi = 2R \times \frac{1}{2\sqrt{2}} \sqrt{(5-\sqrt{5})}$, (p. 61, Ex. 41); $\therefore R = \frac{\sqrt{2}}{\sqrt{(5-\sqrt{5})}} = \frac{\sqrt{2}\sqrt{(5+\sqrt{5})}}{\sqrt{20}} = \sqrt{\left(\frac{5+\sqrt{5}}{10}\right)}$.

4. $2f = a\left(\operatorname{cosec}\frac{\pi}{n} + \cot\frac{\pi}{n}\right) = a\,\dfrac{1+\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}} = a\,\dfrac{2\cos^2\frac{\pi}{2n}}{2\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}} = 2l.$

5. Area $= \pi r^2 = \frac{1}{2} \cdot 2\pi r \cdot r = \frac{1}{2}$ circumf. $\times$ rad.

6. Area $= \frac{1}{4}\pi(2r)^2 = \frac{1}{4}$ of $3{\cdot}1416 \times (2r)^2 = (\text{diameter})^2 \times {\cdot}7854$.

7. Area $= \frac{1}{2}\theta r^2 = \frac{1}{2}r \times \theta r = \frac{1}{2}r$. arc. [See (2), p. 115.]

8. Area of segment = that of sector $-$ that of triangle between chord and radii to its extremities $= \frac{1}{2}\theta r^2 - \frac{1}{2}r \cdot r\sin\theta = \frac{1}{2}r^2(\theta - \sin\theta)$.

9. On making the figure, it is evident that, R, r being the radii, and t the tangent, $t^2 = R^2 - r^2$. Evidently, area between circumferences $= \pi R^2 - \pi r^2 = \pi(R^2 - r^2) = \pi t^2$.

10. If ρ be radius of circle, area of inscribed $\times$ area of circumscribed equilateral triangle $= \frac{3}{2}\rho^2 \sin 120° \times 3\rho^2 \tan 60° = \frac{27}{4}\rho^4$. Also, (area of hexagon$)^2 = (3\rho^2 \sin 60°)^2 = \frac{27}{4}\rho^4$.

11. Let α, β, γ be the sides of pentagon, hexagon, and decagon respectively, and ρ radius; then $\beta^2 + \gamma^2 = (2\rho\sin\frac{1}{6}\pi)^2 + (2\rho\sin\frac{1}{10}\pi)^2 = 4\rho^2\left(\frac{1}{4} + \frac{6-2\sqrt{5}}{16}\right) = 4\rho^2\frac{10-2\sqrt{5}}{16} = 4\rho^2\frac{5-\sqrt{5}}{8} = (2\rho\sin\frac{1}{5}\pi)^2 = \alpha^2$.

12. Suppose α the angle subtended at centre by the side a; then radius $= ma$; $\therefore a = 2ma\sin\frac{1}{2}\alpha$, $\therefore \cos\alpha = 1 - 2\sin^2\frac{1}{2}\alpha = 1 - \frac{2}{4m^2} = \frac{2m^2-1}{2m^2}$, $\therefore \sec\alpha = \frac{2m^2}{2m^2-1}$.

13. Let x and $x+1$ be required numbers. By Euc. I. 32, Cor. 1, $\frac{2x \cdot 90° - 360°}{x} = \frac{2(x+1)90° - 360°}{x+1} - 4°$, $\therefore \frac{45x-90}{x} = \frac{45x-45}{x+1} - 1$, $\therefore 45(x^2 - x - 2) = 45x^2 - 45x - x^2 - x$, $\therefore x^2 + x - 90 = 0$, $\therefore x = 9$ and $x + 1 = 10$.

14. Let ρ, ρ' be the radii; then, by question, $\frac{2}{3}\pi\rho = 2\pi\rho'$, $\therefore \rho : \rho' = 3 : 1$. Let A be no. of degrees reqd., and θ the corresponding circ. meas.; then $\theta = \frac{\rho}{\rho'} = 3$, $\therefore A : 3 = 180° : \pi$, $\therefore A = \frac{180° \times 3}{\pi} = \frac{225000°}{1309} = 171° 53' 14''$.

15. The three radii are as $1\frac{2}{3} : 2\frac{1}{2} : 1\frac{2}{3} + 2\frac{1}{2} = 10 : 15 : 25 = 2 : 3 : 5$. Let them be 2ρ, 3ρ, 5ρ respectively; then the areas are as
$$\pi \,.\, 4\rho^2 : \pi \,.\, 9\rho^2 : \pi \,.\, 25\rho^2 = 4 : 9 : 25.$$

16. Let x = no. of sides, and ρ radius of circle; then, by question,
$$\tfrac{1}{2}x\rho^2 \sin\frac{2\pi}{x} : x\rho^2 \tan\frac{\pi}{x} = 3 : 4, \quad \therefore\ 2\sin\frac{2\pi}{x} = 3\tan\frac{\pi}{x},$$
$$\therefore\ 4\sin\frac{\pi}{x}\cos^2\frac{\pi}{x} = 3\sin\frac{\pi}{x}, \quad \therefore\ \cos\frac{\pi}{x} = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}, \quad \therefore\ x = 6.$$

CHAPTER XX.

1. $\tan\alpha = \dfrac{\sin\alpha}{\sqrt{(1-\sin^2\alpha)}} = \dfrac{s}{\sqrt{(1-s^2)}}$.

2. $\operatorname{cosec}\alpha = \dfrac{1}{\sqrt{(1-\cos^2\alpha)}} = \dfrac{1}{\sqrt{(1-c^2)}}$.

3. $\operatorname{vers}\alpha = 1 - \dfrac{\cos\alpha}{\sin\alpha}\sin\alpha = 1 - \dfrac{\cot\alpha}{\sqrt{(\cot\alpha+1)}} = 1 - \dfrac{t}{\sqrt{(t^2+1)}} = \dfrac{\sqrt{(t^2+1)}-t}{\sqrt{(t^2+1)}}$.

4. If $\alpha = \cos^{-1}x$, or $\cos\alpha = x$, then $\sec\alpha = \dfrac{1}{x}$, or $\alpha = \sec^{-1}\dfrac{1}{x}$.

5. If $\alpha = \sin^{-1}x$, or $\sin\alpha = x$, then $\cos\alpha = \sqrt{(1-x^2)}$, or $\alpha = \cos^{-1}\sqrt{(1-x^2)}$.

6. If $\alpha = \cot^{-1}x$ or $\cot\alpha = x$, then $\operatorname{cosec}\alpha = \sqrt{(x^2+1)}$, or $\alpha = \operatorname{cosec}^{-1}\sqrt{(x^2+1)}$.

7. Let $\alpha = \sec^{-1}x$, or $\sec\alpha = x$, then $\tan\alpha = \sqrt{(x^2-1)}$.

8. Let $\alpha = \tan^{-1}\dfrac{b}{a}$, or $\tan\alpha = \dfrac{b}{a}$, then $a\sec\alpha = a\sqrt{\left(\dfrac{b^2}{a^2}+1\right)} = \sqrt{(a^2+b^2)}$.

9. Let $\alpha = \sin^{-1}\dfrac{b}{a}$, or $\sin\alpha = \dfrac{b}{a}$, then $a\cos\alpha = a\sqrt{\left(1-\dfrac{b^2}{a^2}\right)} = \sqrt{(a^2-b^2)}$.

10. Let $\alpha = \sin^{-1}\sqrt{[\frac{1}{2}(1-x)]}$, or $\sin\alpha = \sqrt{[\frac{1}{2}(1-x)]}$, then $\cos 2\alpha = 1 - 2 \,.\, \frac{1}{2}(1-x) = x$, or $2\alpha = \cos^{-1}x$.

11. Let $\alpha = \sin^{-1}x$, or $\sin\alpha = x$; then $\sin 2\alpha = 2\sin\alpha\cos\alpha = 2x\sqrt{(1-x^2)}$.

12. Let $\alpha = \tan^{-1}\sqrt{\dfrac{x}{a}}$, or $\tan\alpha = \sqrt{\dfrac{x}{a}}$; $\cos 2\alpha = 1 - 2\dfrac{\sin^2\alpha}{\cos^2\alpha}\cos^2\alpha =$
$$1 - \frac{2\tan^2\alpha}{\tan^2\alpha+1} = \frac{1-\tan^2\alpha}{1+\tan^2\alpha} = \frac{1-\dfrac{x}{a}}{1+\dfrac{x}{a}} = \frac{a-x}{a+x}, \text{ or } 2\alpha = \cos^{-1}\frac{a-x}{a+x}.$$

13. $2\tan^{-1}\frac{1}{3} = \tan^{-1}\dfrac{2\,.\,\frac{1}{3}}{1-\frac{1}{9}} = \tan^{-1}\frac{3}{4} = \cot^{-1}\frac{4}{3}$.

14. $2\tan^{-1}\frac{2}{3} = \tan^{-1}\dfrac{2\,.\,\frac{2}{3}}{1-\frac{4}{9}} = \tan^{-1}\frac{12}{5}$; $\therefore\ \tan^{-1}\frac{2}{3} = \frac{1}{2}\tan^{-1}\frac{12}{5}$.

15. $\sin\cot^{-1}x = \sin\operatorname{cosec}^{-1}\sqrt{(x^2+1)} = \dfrac{1}{\sqrt{(x^2+1)}}$, and

$$\cos\tan^{-1}\frac{1}{\sqrt{(x^2+1)}} = \cos\sec^{-1}\sqrt{\left(\frac{1}{x^2+1}+1\right)}$$
$$= \cos\sec^{-1}\sqrt{\left(\frac{x^2+2}{x^2+1}\right)} = \sqrt{\left(\frac{x^2+1}{x^2+2}\right)}.$$

16. $f = 30^\circ + 45^\circ = l$. 17. Let $\alpha = \tan^{-1}t_1$, $\beta = \tan^{-1}t_2$, then $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \dfrac{t_1+t_2}{1-t_1t_2}$.

18. $$f = \tan^{-1}\frac{2+\sqrt{3}-\dfrac{1}{\sqrt{3}}}{1+(2+\sqrt{3})\dfrac{1}{\sqrt{3}}} = \tan^{-1}\frac{2+2\sqrt{3}}{2+2\sqrt{3}} = \tan^{-1}1 = l.$$

19. $f = \tan^{-1}\frac{3}{5}+\tan^{-1}\frac{3}{7} = \tan^{-1}\dfrac{\frac{3}{5}+\frac{3}{7}}{1-\frac{3}{5}\cdot\frac{3}{7}} = \tan^{-1}\frac{36}{26} = \tan^{-1}\frac{18}{13} = l.$

20. $f = \tan\tan^{-1}\dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}} = \tan\tan^{-1}1 = l.$

21. $f = \cot\tan^{-1}\dfrac{\frac{4}{3}+7}{1-\frac{4}{3}\cdot\frac{7}{1}} = \cot\tan^{-1}(-1) = \cot\cot^{-1}(-1) = l.$

22. $$f = \tan^{-1}\left\{\left(\frac{2x-y}{y\sqrt{3}}+\frac{2y-x}{x\sqrt{3}}\right) \div \left(1-\frac{(2x-y)(2y-x)}{3xy}\right)\right\}$$
$$= \tan^{-1}\left\{\frac{2(x^2-xy+y^2)}{xy\sqrt{3}} \times \frac{3xy}{2(x^2-xy+y^2)}\right\} = \tan^{-1}\sqrt{3} = \tfrac{1}{3}\pi.$$

23. $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13} = \sin^{-1}\left[\frac{4}{5}\sqrt{(1-\frac{25}{169})}+\frac{5}{13}\sqrt{(1-\frac{16}{25})}\right]$
$= \sin^{-1}\left[\frac{4}{5}\cdot\frac{12}{13}+\frac{5}{13}\cdot\frac{3}{5}\right] = \sin^{-1}\frac{63}{65} = \cos^{-1}\sqrt{[1-(\frac{63}{65})^2]} = \cos^{-1}\frac{16}{65}$
$= \frac{1}{2}\pi - \sin^{-1}\frac{16}{65}$; transposing, $f = l$.

24. $$f = \tan^{-1}\frac{\frac{1}{4}+\frac{1}{8}}{1-\frac{1}{4}\cdot\frac{1}{8}}+\tan^{-1}\frac{\frac{2}{9}+\frac{1}{5}}{1-\frac{2}{9}\cdot\frac{1}{5}} = \tan^{-1}\tfrac{12}{31}+\tan^{-1}\tfrac{19}{43}$$
$$= \tan^{-1}\frac{\frac{12}{31}+\frac{19}{43}}{1-\frac{12}{31}\cdot\frac{19}{43}} = \tan^{-1}1 = \tfrac{1}{4}\pi.$$

25. $$f = \tan(3\tan^{-1}x) = \tan\left(\tan^{-1}x+\tan^{-1}\frac{2x}{1-x^2}\right)$$
$$= \left(x+\frac{2x}{1-x^2}\right) \div \left(1-\frac{x\,.\,2x}{1-x^2}\right) = \frac{3x-x^3}{1-x^2}\times\frac{1-x^2}{1-3x^2} = l.$$

26. $f = \tan\tan^{-1}\dfrac{2x}{1-x^2} = \dfrac{2x}{1-x^2}$; and $l = 2\tan\tan^{-1}\dfrac{x+x^3}{1-x^4} =$
$$2\frac{x(1+x^2)}{1-x^4} = \frac{2x}{1-x^2};\quad \therefore f = l.$$

27. Let $\alpha = \sin^{-1}\dfrac{2\sqrt{(ac)}}{b}$; $\cos\alpha = \sqrt{\left(1-\dfrac{4ac}{b^2}\right)} = \dfrac{\sqrt{(b^2-4ac)}}{b}$,
$$\therefore f = -\frac{b}{a}\,.\,\tfrac{1}{2}(1-\cos\alpha) = -\frac{b}{2a}\left(1-\frac{\sqrt{(b^2-4ac)}}{b}\right) = l.$$

28. Let $\alpha = \sin^{-1}\dfrac{2\sqrt{(ac)}}{b}$, then $\cos\alpha = \dfrac{\sqrt{(b^2-4ac)}}{b}$;

$\therefore\ -\dfrac{b}{a}\cdot\dfrac{1}{2}\,2\cos^2\tfrac{1}{2}\alpha = -\dfrac{b}{2a}(1+\cos\alpha) = -\dfrac{b}{2a}\left\{1+\dfrac{\sqrt{(b^2-4ac)}}{b}\right\} = l.$

29. $f = (a+b)\sin\tfrac{1}{2}C\sqrt{\left\{1+\dfrac{(a-b)^2\cos^2\frac{1}{2}C}{(a+b)^2\sin^2\frac{1}{2}C}\right\}}$
$= \sqrt{[(a+b)^2\sin^2\tfrac{1}{2}C+(a-b)^2\cos^2\tfrac{1}{2}C]} = \sqrt{[(a^2+b^2)(\cos^2\tfrac{1}{2}C+\sin^2\tfrac{1}{2}C)}$
$-2ab(\cos^2\tfrac{1}{2}C-\sin^2\tfrac{1}{2}C)] = \sqrt{[(a^2+b^2)-2ab\cos C]} = c.$

30. $\sin^{-1}(\tan\tfrac{1}{4}\pi) = \sin^{-1}1$ or $\tfrac{1}{2}\pi$; and $\tan\sin^{-1}\dfrac{1}{\sqrt{2}} = \tan\tfrac{1}{4}\pi = 1.$

31. $\tan^{-1}\dfrac{2x+3x}{1-2x\,.\,3x} = \tfrac{1}{4}\pi = \tan^{-1}$; or $\dfrac{5x}{1-6x^2} = 1$; or $6x^2+5x-1=0$,

$\therefore\ x = \dfrac{-5\pm\sqrt{(25+24)}}{12} = \tfrac{1}{6}$ or $-1.$

32. $\tan^{-1}\dfrac{x-\dfrac{1}{x}}{1+x\dfrac{1}{x}} = \pi = \tan^{-1}0$; $x^2-1=0$, $\therefore\ x = \pm 1.$

33. $\sin^{-1}\dfrac{2a}{1+a^2} = \operatorname{cosec}^{-1}\dfrac{1+a^2}{2a} = \cot^{-1}\sqrt{\left\{\dfrac{(1+a^2)^2}{4a}-1\right\}}$

$= \cot^{-1}\dfrac{1-a^2}{2a} = \tan^{-1}\dfrac{2a}{1-a^2} = 2\tan^{-1}a$; $\therefore\ 2\tan^{-1}a+2\tan^{-1}b=$

$2\tan^{-1}x$, $\therefore\ \tan^{-1}x = \tan^{-1}\dfrac{a+b}{1-ab}$, $\therefore\ x = \dfrac{a+b}{1-ab}.$

34. $\cot^{-1}\dfrac{1}{x+1}+\cot^{-1}\dfrac{1}{x-1} = \tan^{-1}\dfrac{x+1+x-1}{1-(x+1)(x-1)} = \tan^{-1}\dfrac{2x}{2-x^2}$;

and $\tan^{-1}3x-\cot^{-1}\dfrac{1}{x} = \tan^{-1}\dfrac{3x-x}{1+3x^2} = \tan^{-1}\dfrac{2x}{1+3x^2}$,

$\therefore\ \dfrac{2x}{2-x^2} = \dfrac{2x}{1+3x^2}$, $\therefore\ x=0$, or $4x^2=1$, and $x = \pm\tfrac{1}{2}.$

35. $\sec^{-1}a+\sec^{-1}\dfrac{x}{a} = \cos^{-1}\dfrac{1}{a}+\cos^{-1}\dfrac{a}{x}$

$= \cos^{-1}\left(\dfrac{a}{ax}-\dfrac{\sqrt{[(a^2-1)(x^2-a^2)]}}{ax}\right)$, $\sec^{-1}b+\sec^{-1}\dfrac{x}{b} =$ &c.,

$\therefore\ \dfrac{1}{x}-\dfrac{\sqrt{[(a^2-1)(x^2-a^2)]}}{ax} = \dfrac{1}{x}-\dfrac{\sqrt{[(b^2-1)(x^2-b^2)]}}{bx}$,

$\therefore\ b^2(a^2-1)(x^2-a^2) = a^2(b^2-1)(x^2-b^2)$,

$\therefore\ x^2(a^2-b^2) = a^2b^2(a^2-b^2)$, $\therefore\ x = \pm ab.$

36. $\cos^{-1}(-x)-\cos^{-1}x = \cos^{-1}[-x^2+1-x^2] = \sec^{-1}\dfrac{1}{1-2x^2}$

$= \tan^{-1}\dfrac{\sqrt{(4x^2-4x^4)}}{1-2x^2}$, $\therefore\ 2\sqrt{(1-x^2)} = \dfrac{2x\sqrt{(1-x^2)}}{1-2x^2}$,

$\therefore$ (1) $x^2-1=0$, and $x=\pm 1$; (2) $2x^2+x-1=0$,

$\therefore\ x = \dfrac{-1\pm\sqrt{9}}{4} = \tfrac{1}{2}$ or $-1.$

CHAPTER XXI.

1. $4\tan^2\alpha = 3\sec^2\alpha = 3(\tan^2\alpha+1)$; $\therefore \tan^2\alpha = 3$; $\therefore \tan\alpha = \pm\sqrt{3}$ $= \tan\frac{1}{3}\pi$ or $\tan -\frac{1}{3}\pi$, $\therefore \alpha = n\pi \pm \frac{1}{3}\pi$.

2. $4\cos^2 A = 3$, $\therefore \cos A = \pm\frac{1}{2}\sqrt{3} = \cos 30°$ or $\cos 150°$, $\therefore A = 2n\,.\,180° \pm 30°$ or $2n\,.\,180° \pm 150° = n\,.\,180° \pm 30°$.

3. $\cos A = \sqrt{2}\cos A \sin A$, $\therefore \cos A(\sqrt{2}\sin A - 1) = 0$;
$\therefore$ (1) $\cos A = 0 = \cos 90°$, $\therefore A = 2n\,.\,180° \pm 90° = n\,.\,180° + 90°$;

$$(2)\ \sin A = \frac{1}{\sqrt{2}} = \sin 45°, \quad \therefore A = n\,.\,180° + (-1)^n 45°.$$

4. $2 - \cos\theta = \sqrt{3}\sin\theta$, $\therefore 4 - 4\cos\theta + \cos^2\theta = 3\sin^2\theta = 3(1-\cos^2\theta)$,
$\therefore 4\cos^2\theta - 4\cos\theta + 1 = 0$, or $(2\cos\theta - 1)^2 = 0$,
$\therefore \cos\theta = \frac{1}{2} = \cos\frac{1}{3}\pi$, $\therefore \theta = 2n\pi \pm \frac{1}{3}\pi$.

5. $\cos^2\theta + 4\cos^2\theta\sin^2\theta = 6\sin^2\theta$, or $\cos^2\theta + 4\cos^2\theta(1-\cos^2\theta) = 6(1-\cos^2\theta)$,
$\therefore 4\cos^4\theta - 11\cos^2\theta + 6 = 0$, $\therefore \cos^2\theta = \frac{1}{8}(11 \pm \sqrt{25}) = 2$ (impos.) or $\frac{3}{4}$, $\therefore \cos\theta = \pm\frac{1}{2}\sqrt{3}$; $\therefore$ [as in 2] $\theta = n\pi \pm \frac{1}{6}\pi$.

6. $\sec\theta = \sec\pi$, $\therefore \theta = 2n\pi \pm \pi$, *i.e.* $(2n+1)\pi$.

7. $\cos^2\alpha = 1$, $\cos\alpha = \pm 1 = \cos 0$ or π, $\alpha = 2n\pi$ or $(2n+1)\pi$, *i.e.* $n\pi$.

8. $\cot\theta(\cot\theta + \sqrt{3}) = 0$; $\therefore$ (1) $\cot\theta = 0$, $\theta = n\pi + \frac{1}{2}\pi$;
(2) $\cot\theta = -\sqrt{3} = \cot\frac{5}{6}\pi$, $\theta = n\pi + \frac{5}{6}\pi$.

9. $\sin\theta = \frac{1}{4}(-5 \pm \sqrt{9}) = -\frac{1}{2}$ or -2 (impos.); $\therefore \theta = n\pi + (-1)^n\frac{7}{6}\pi$.

10. $\sin\theta(2\cos\theta + 1) = 0$; $\therefore$ (1) $\sin\theta = 0 = \sin 0$, $\therefore \theta = n\pi$;
(2) $\cos\theta = -\frac{1}{2} = \cos\frac{2}{3}\pi$, $\therefore \theta = 2n\pi \pm \frac{2}{3}\pi$.

11. $2\sin 2\theta\cos\theta + \sin 2\theta = 0$, or $\sin 2\theta(2\cos\theta + 1) = 0$;
$\therefore$ (1) $\sin 2\theta = 0$, $\therefore 2\theta = n\pi$. (2) As (2) in 10.

12. $2\cos 2\theta\cos\theta + \cos 2\theta = 0$, or $\cos 2\theta(2\cos\theta + 1) = 0$;
$\therefore$ (1) $\cos 2\theta = 0$, $\therefore 2\theta = 2n\pi \pm \frac{1}{2}\pi = n\pi + \frac{1}{2}\pi$. (2) As (2) in 10.

13. $2\cos\frac{3}{4}\theta\cos\frac{1}{4}\pi - \cos\frac{3}{4}\theta = 0$, or $\cos\frac{3}{4}\theta(2\cos\frac{1}{4}\theta - 1) = 0$;
(1) $\cos\frac{3}{4}\theta = 0$, $\therefore \frac{3}{4}\theta = n\pi + \frac{1}{2}\pi$ [as (1) in 3];
(2) $\cos\frac{1}{4}\theta = \frac{1}{2}$, $\therefore \frac{1}{4}\theta = 2n\pi \pm \frac{1}{3}\pi$.

14. $2\cos 4\theta\sin 3\theta - \sin 3\theta = 0$, $\therefore \sin 3\theta(2\cos 4\theta - 1) = 0$;
$\therefore$ (1) $\sin 3\theta = 0$, $3\theta = n\pi$; (2) $\cos 4\theta = \frac{1}{2}$, $4\theta = 2n\pi \pm \frac{1}{3}\pi$.

15. $\cos\theta\dfrac{1}{\sqrt{2}} + \sin\theta\dfrac{1}{\sqrt{2}} = \frac{1}{2}$, or $\cos(\theta - \frac{1}{4}\pi) = \frac{1}{2}$, $\theta - \frac{1}{4}\pi = 2n\pi \pm \frac{1}{3}\pi$.

16. $\sin\theta\dfrac{1}{\sqrt{2}} + \cos\theta\dfrac{1}{\sqrt{2}} = 1$, or $\sin(\theta + \frac{1}{4}\pi) = 1$, $\theta + \frac{1}{4}\pi = 2n\pi + \frac{1}{2}\pi$.

17. $\sin\theta\dfrac{\sqrt{3}}{2} - \cos\theta\dfrac{1}{2} = \dfrac{1}{\sqrt{2}}$, or $\sin(\theta - \frac{1}{6}\pi) = \dfrac{1}{\sqrt{2}}$, $\theta - \frac{1}{6}\pi = n\pi + (-1)^n\frac{1}{4}\pi$.

18. $\cos\theta\dfrac{1}{\sqrt{2}} - \sin\theta\dfrac{1}{\sqrt{2}} = 1$, or $\cos(\theta + \frac{1}{4}\pi) = 1$, $\theta + \frac{1}{4}\pi = 2n\pi$.

19. $\tan(\frac{1}{4}\pi + \alpha)\tan(\frac{1}{4}\pi + \alpha) = 3$, $\therefore \tan(\frac{1}{4}\pi + \alpha) = \pm\sqrt{3}$, $\frac{1}{4}\pi + \alpha = n\pi \pm \frac{1}{3}\pi$.

20. $2\cos(n-1)\theta\cos\theta = \cos\theta$, or $\cos\theta[2\cos(n-1)\theta - 1] = 0$;

$\therefore$ (1) $\cos\theta = 0$, $\theta = n\pi + \frac{1}{2}\pi$; (2) $\cos(n-1)\theta = \frac{1}{2}$, $(n-1)\theta = 2n\pi \pm \frac{1}{3}\pi$.

21. $\sec\theta = \frac{1}{4}[-2\sqrt{2}-1 \pm \sqrt{(9+4\sqrt{2}-8\sqrt{2})}] = -\sqrt{2}$ or $-\frac{1}{2}$(impos.); $\theta = 2n\pi \pm \frac{3}{4}\pi$.

22. $\cos 4a + \cos 2a = \cos 12a + \cos 2a$, $\therefore$ $2\sin 8a \sin 4a = 0$;
$\therefore$ (1) $\sin 8a = 0$, $\therefore$ $8a = n\pi$; (2) $\sin 4a = 0$, $4a = n\pi$.

23. $\cos a = \sin a - 4\sin a(1-\sin^2 a) = 4\sin^3 a - 3\sin a = -\sin 3a$;
$\therefore$ $\cos(\frac{1}{2}\pi + 3a) = \cos a$, $\therefore$ $\frac{1}{2}\pi + 3a = 2n\pi \pm a$.

24. $\sin a - \cos a = 2\sqrt{2}\sin a\cos a = \sqrt{2}\sin 2a$, $\therefore$ $1-\sin 2a = 2\sin^2 2a$,
$\therefore$ $2\sin^2 2a + \sin 2a - 1 = 0$, $\therefore$ $\sin 2a = \frac{1}{4}(-1\pm\sqrt{9}) = \frac{1}{2}$ or -1,
$\therefore$ $2a = n\pi + (-1)^n \frac{1}{6}\pi$ or $2n\pi - \frac{1}{2}\pi$.

CHAPTER XXII.

1. Since 300° is in 4th quad., in which the tangent is negative,
$\tan 300° = -\sqrt{\sec^2 300° - 1} = -\sqrt{4-1} = -\sqrt{3}$.

2. Since $\tan A = \tan(n \,.\, 180° + A)$, $\sqrt{\tan^2 A + 1}$ must give, for all values of n, $\sec(n \,.\, 180° + A)$.

Geom.—But $n \,.\, 180° + A$, when n is even, gives one group of angles; when n is odd, another; and (fig., p. 126) if $AOP = A$, the one group is bounded by OA and OP, the other by OA and OP'. The two groups are thus in vertically opposite quadrants; and their secants therefore differ in sign.

Analyt.—(1) If n be even, and $= 2m$ suppose, then
$\sec(n \,.\, 180° + A) = \sec(m \,.\, 360° + A) = \sec A$.
(2) If n be odd, and $= 2m+1$ suppose, then
$\sec(n \,.\, 180° + A) = \sec(m \,.\, 360° + 180° + A) = \sec(180° + A) = -\sec A$.

3. $f = \frac{1}{2}\sqrt{2+2\cos\frac{1}{8}\pi} = \frac{1}{2}\sqrt{2+2 \,.\, \frac{1}{2}\sqrt{2+2\cos\frac{1}{4}\pi}} = l$.

4. $f = \frac{1}{2}\sqrt{2-2\cos\frac{1}{16}\pi} = \frac{1}{2}\sqrt{2-2 \,.\, \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}} = l$.

5. $f = \frac{1}{2}\sqrt{2+2\cos\frac{1}{12}\pi} = \frac{1}{2}\sqrt{2+2 \,.\, \frac{1}{2}\sqrt{2+2\cos\frac{1}{6}\pi}} = l$.

6. $f = \frac{1}{2}\sqrt{2-2\cos\frac{1}{24}\pi} = \frac{1}{2}\sqrt{2-2 \,.\, \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{3}}}} = l$.

7. $f = \frac{1}{2}\sqrt{2+2\cos\frac{1}{10}\pi} = \frac{1}{2}\sqrt{\left(2+2\frac{1}{2\sqrt{2}}\sqrt{5+\sqrt{5}}\right)} = l$.

8. $f = \frac{1}{2}\sqrt{2-2\cos\frac{1}{20}\pi} = \frac{1}{2}\sqrt{2-2 \,.\, \frac{1}{2}\sqrt{2+\sqrt{\frac{1}{2}(5+\sqrt{5})}}} = l$.

9. Divide a revolution into eight semi-quadrants. Now the limits $n\pi + \frac{1}{4}\pi$ and $n\pi + \frac{3}{4}\pi$, when n is even, embrace the 2nd and 3rd semi-quads.; when n is odd, the 6th and 7th. Here (fig., p. 41) $PN > ON$, and $\therefore$ sine > cosine. $n\pi - \frac{1}{4}\pi$ and $n\pi + \frac{1}{4}\pi$, when n is even, embrace the 8th and 1st; when n is odd, the 4th and 5th. Here $ON > PN$, and $\therefore$ cosine > sine.

10. $\sin A - \cos A = \sqrt{2}\sin(A - 45°)$; and, *in sign*, is +ve, when $A - 45°$ is between 0 and 180°, and $\therefore$ when A is between 45° and 225°; −ve, when $A - 46°$ is between 180° and 360°, and $\therefore$ for A from 225° to 360°, and 0 to 45°.

In magnitude, as A increases positively, it $= 0$ when $A = 45°$, increases to 1 when $A = 135°$, decreases to 0 when $A = 225°$, increases to -1 when $A = 315°$, decreases to 0 at $405°$.

11. $\tan A + \cot A = \dfrac{\sin^2 A + \cos^2 A}{\sin A \cos A} = \dfrac{2}{\sin 2A} = 2 \operatorname{cosec} 2A$; and, *in sign*, is +ve, when $2A$ is between 0 and 180°, and $\therefore$ when A is between 0 and 90°; −ve, when $2A$ is between 180° and 360°, and $\therefore$ when A is between 90° and 180°.

In magnitude, as A increases positively, it $= \infty$ when $A = 0$, decreases to 1 when $A = 45°$, increases to ∞ when $A = 90°$, decreases to -1 when $A = 135°$, increases to ∞ when $A = 180°$.

The variation when A is in 3rd and 4th quads. repeats the above.

12. $\cos 15° > \sin 15°$ and +ve, $\therefore \sin 15° + \cos 15° = \sqrt{(1 + \sin 30°)} = \sqrt{(1 + \frac{1}{2})} = \sqrt{\frac{3}{2}}$; and $\sin 15° - \cos 15° = -\sqrt{(1 - \sin 30°)} = -\sqrt{\frac{1}{2}}$. Adding, $2 \sin 15° = \dfrac{\sqrt{3} - 1}{\sqrt{2}}$, $\therefore \sin 15° = \dfrac{\sqrt{3} - 1}{2\sqrt{2}}$.

13. $\sin 75° > \cos 75°$ and +ve, $\therefore$ $75° + \cos 75° = \sqrt{(1 + \sin 150°)} = \sqrt{\frac{3}{2}}$, and $\sin 75° - \cos 75° = \sqrt{(1 - \sin 30°)} = \sqrt{\frac{1}{2}}$. Adding, $2 \sin 75° = \dfrac{\sqrt{3} + 1}{2}$, $\therefore \sin 75° = \dfrac{\sqrt{3} + 1}{2\sqrt{2}}$.

14. $\sin(-105°) > \cos(-105°)$ and −ve, $\therefore \sin(-105°) + \cos(-105°) = -\sqrt{[1 + \sin(-210°)]} = -\sqrt{(1 + \frac{1}{2})} = \dfrac{-\sqrt{3}}{\sqrt{2}}$, and $\sin(-105°) - \cos(-105°) = -\sqrt{[1 - \sin(-210°)]} = -\dfrac{1}{\sqrt{2}}$. Add and divide, $\sin(-105°) = \dfrac{-\sqrt{3} - 1}{2\sqrt{2}}$.

15. $\cos(-165°) > \sin(-165°)$ and −ve, $\therefore \sin(-165°) + \cos(-165°) = -\sqrt{[1 + \sin(-330°)]} = -\sqrt{\frac{3}{2}}$, and $\sin(-165°) - \cos(-165°) = \sqrt{[1 - \sin(-330°)]} = \sqrt{\frac{1}{2}}$. Add and divide, $\sin(-165°) = \dfrac{-\sqrt{3} + 1}{2\sqrt{2}}$.

16. $\cos(-15°) > \sin(-15°)$ and +ve, $\therefore \sin(-15°) + \cos(-15°) = \sqrt{[1 + \sin(-30°)]} = \sqrt{(1 - \frac{1}{2})} = \sqrt{\frac{1}{2}}$, and $\sin(-15°) - \cos(-15°) = -\sqrt{[1 - \sin(-30°)]} = -\sqrt{(1 + \frac{1}{2})} = -\sqrt{\frac{3}{2}}$. Subtract and divide, $\cos(-15°) = \dfrac{1 + \sqrt{3}}{2\sqrt{2}}$.

17. $\sin(-75°) > \cos(-75°)$ and −ve, $\therefore \sin(-75°) + \cos(-75°) = -\sqrt{[1 + \sin(-150°)]} = -\sqrt{(1 - \frac{1}{2})} = -\sqrt{\frac{1}{2}}$, and $\sin(-75°) - \cos(-75°) = -\sqrt{[1 - \sin(-150°)]} = -\sqrt{(1 + \frac{1}{2})} = -\sqrt{\frac{3}{2}}$. Subt. and div., $\cos(-75°) = \dfrac{-1 + \sqrt{3}}{2\sqrt{2}}$.

18. $\sin 105° > \cos 105°$ and +ve, $\therefore \sin 105° + \cos 105° = \sqrt{(1 + \sin 210°)} = \sqrt{(1 - \frac{1}{2})} = \dfrac{1}{\sqrt{2}}$, and $\sin 105° - \cos 105° = \sqrt{(1 - \sin 210°)} = \sqrt{(1 + \frac{1}{2})} = \sqrt{\frac{3}{2}}$. Subt. and div., $\cos 105° = \dfrac{1 - \sqrt{3}}{2\sqrt{2}}$.

19. $\cos 165° > \sin 165°$ and $-$ve, $\therefore \sin 165° + \cos 165° = -\sqrt{(1+\sin 330°)} = -\sqrt{(1-\frac{1}{2})} = \frac{-1}{\sqrt{2}}$, and $\sin 165° - \cos 165° = \sqrt{(1-\sin 330°)} = \sqrt{(1+\frac{1}{2})} = \sqrt{\frac{3}{2}}$. Subt. and div., $\cos 165° = \frac{-1-\sqrt{3}}{2\sqrt{2}}$.

20. $\cos 9° > \sin 9°$ and $+$ve, $\therefore \sin 9° + \cos 9° = \sqrt{(1+\sin 18°)} = \sqrt{[1+\frac{1}{4}(\sqrt{5}-1)]} = \frac{1}{2}\sqrt{(3+\sqrt{5})}$, and $\sin 9° - \cos 9° = -\sqrt{(1-\sin 18°)} = -\sqrt{[1-\frac{1}{4}(\sqrt{5}-1)]} = -\frac{1}{2}\sqrt{(5-\sqrt{5})}$. Add and divide, $\sin 9° = \frac{1}{4}[\sqrt{(3+\sqrt{5})} - \sqrt{(5-\sqrt{5})}]$.

21. $\frac{1}{2}A$ is between 90° and 135°, $\therefore \sin\frac{1}{2}A > \cos\frac{1}{2}A$ and $+$ve, $\therefore \sin\frac{1}{2}A + \cos\frac{1}{2}A = \sqrt{(1+\sin A)} = \sqrt{\frac{6}{5}}$, and $\sin\frac{1}{2}A - \cos\frac{1}{2}A = \sqrt{(1-\sin A)} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$. Subtract and divide, $\cos\frac{1}{2}A = \frac{\sqrt{6}-2}{2\sqrt{5}} = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{10}}$.

EXAMINATION PAPERS.

I.

1. $25° - 25^g = 25° - \frac{9}{10}25° = \frac{1}{10}25° = 2\frac{1}{2}°$. $25° - 25^g = \frac{10}{9}25^g - 25^g = \frac{1}{9}25^g = 2\frac{7}{9}{}^g$.

2. Since a right angle in the respective measures $= 90° = 100^g = \frac{1}{2}\pi$, $\therefore G : D = 100 : 90 = 10 : 9$, and $G : 20C = 100 : 20\frac{1}{2}\pi = 10 : \pi$; $\therefore \frac{G}{10} = \frac{D}{9}$, and also $= \frac{20C}{\pi}$.

3. $3(\sin^2\frac{1}{2}A + \cos^2\frac{1}{2}A) = 3 \times 1 = 3$.

4. $2 - v = 1 + 1 - v = 1 + \cos A$, $\therefore \sin^2 A = 1 - \cos^2 A = (1-\cos A)(1+\cos A) = v(2-v)$, $\therefore f = l$.

5. In fig., p. 17, let $A = B = 45°$ (Euc. I. 32); then $AC = BC$ (Euc. I. 6); $\therefore \frac{BC}{AC} = 1$, or $\tan A = \cot B = 1$.

6. $\text{covers}(90° - A) = 1 - \sin(90° - A) = 1 - \cos A = \text{vers } A$.

II.

1. vertical angle $= 2(90° - 87° 15' 13''\cdot95) = 2 \times 2° 44' 46''\cdot05 = 5° 29' 32''\cdot1 = 5°\cdot49225 = \frac{10}{9}$ of $5^g\cdot49225 = 6^g\cdot1025 = 6^g 10' 25''$.

2. Circ. meas. of angle $= \frac{30}{25 \times 12} = \frac{1}{10}$; then $\frac{1}{2}\pi : \frac{1}{10} :: 90° :$ required sexag. meas. $= \frac{90° \times 2}{3\cdot1416 \times 10} = \frac{180°}{31\cdot416} = 5\frac{955}{1309}°$.

3. $f = 1 \times 1$, and $l = 1$; $\therefore f = l$.

4. Constructing as on p. 22, $BA = BC = 2BD$, $\therefore \frac{BA}{BD} = 2$, $\therefore \sec 60° = 2$.

5. $f = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) \div \left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right) = \frac{1+\sqrt{3}}{2} \times \frac{2}{1-\sqrt{3}} = \frac{1+\sqrt{3}}{1-\sqrt{3}} = l$.

6. In fig. on p. 22, let $BA = AC = a$, and AD be p; then $2AD = BA\sqrt{3}$, $\therefore \frac{AD}{BA} = \frac{\sqrt{3}}{2}$, or $\sin B = \frac{\sqrt{3}}{2} = \sin 60°$, $\therefore B = C = 60°$.

III.

1. Angle $= -360° - \frac{26}{30} . 360° = -360° - 312° = -672°$. For circ. meas., $90° : -672° = \frac{\pi}{2} : -\frac{\pi \times 672}{2 \times 90} = -\frac{56}{15}\pi = -\frac{3{\cdot}1416 \times 56}{15} = -11{\cdot}72864$.

2. $l = \text{vers}\, A\,(1 = \cos A) = (1 - \cos A)(1 + \cos A) - 1 - \cos^2 A = f$.

3. $\cot A = \frac{a}{x}$, $\therefore \operatorname{cosec} A = \sqrt{\left(1 + \frac{a^2}{x^2}\right)} = \frac{\sqrt{(x^2 + a^2)}}{x}$, $\therefore \sin A = \frac{x}{\sqrt{(a^2 + x^2)}}$.

4. $f = \sin^2 a\,(1 - \cos^2 \beta) + \cos^2 a\,(1 - \sin^2 \beta) + \sin^2 a \cos^2 \beta + \cos^2 a \sin^2 \beta$
$= \sin^2 a + \cos^2 a = 1$.

5. $\tan(A + B) = \sqrt{3} = \tan 60°$, $\therefore A + B = 60°$; similarly, $A - B = 30°$. Adding, $2A = 90°$; subtracting, $2B = 30°$; $\therefore A = 45°$, $B = 15°$.

6. Let A, B (fig. 30) be the two objects, D, C the successive points of observation; reqd. AB. Given $CD = 200$ yds., $ADC = 67° 30'$, $ADB = 7° 30'$; $\therefore BDC = 60°$. Now $AC = CD \tan ADC = CD \times 2{\cdot}414$, and $BC = CD \tan 60° = CD \times 1{\cdot}732$, $\therefore AB = CD\,(2{\cdot}414 - 1{\cdot}732)$ $= 200$ yds. $\times {\cdot}682 = 136{\cdot}4$ yds.

IV.

1. A straight line is the shortest distance between two points; therefore, if A, B be joined in fig. of Art. 7, arc $AB >$ chord AB. Now let $O = 60°$; then (Euc. I. 32) the triangle ABC would be equiangular, and $\therefore$ chord $AB = AO =$ radius; $\therefore$ arc $AB >$ radius; $\therefore$ angle $O >$ unit of circ. meas.; or $60° >$ unit of circ. meas.

2. $\sin^2 \theta = p^2 - 2pq + q^2$, $4pq = \cos^2 \theta$;
$\therefore 1 = \sin^2 \theta + \cos^2 \theta = p^2 + 2pq + q^2 = (p + q)^2$; $\therefore f = l$.

3. $f = \frac{\sin^2 A \sin^2 B - \cos^2 A \cos^2 B}{\cos^2 A \cos^2 B} = \frac{\sin^2 A\,(1 - \cos^2 B) - (1 - \sin^2 A)\cos^2 B}{\cos^2 A \cos^2 B}$
$= l$.

4. Let A, B be the acute angles, and $B > A$. Then complement of B $= A$, comp. of $90° = 0$; $\therefore 0$, A, and $90° - A$ are in A. P.; $\therefore 2A = 0 + 90° - A$; $\therefore A = 30°$, $\therefore B = 60°$.

5. Divide 1st by 2nd eq.; then $\cos A = \frac{\sqrt{2}}{\sqrt{3}} \cos B$; $\therefore \cos^2 A = \frac{2}{3} \cos^2 B$,

and $\sin^2 A = 2\sin^2 B$. Adding, $1 = \frac{2}{3}\cos^2 B + 2\sin^2 B = -\frac{4}{3}\cos^2 B + 2$, $\therefore \cos B = \frac{\sqrt{3}}{2}$, and $\cos A = \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{2}}$; $\therefore B = 30°$, $A = 45°$.

6. In fig. (8), let BC, DA be the towers. Given FBD ($= BDE$) $= 30°$, $DE = 30$ yds., $BC = 100$ yds. Now $BE = DE \tan 30° = 30 \text{ yds.} \times \frac{1}{\sqrt{3}}$ $= 10\sqrt{3}$ yds. $= 17{\cdot}32$ yds.; $\therefore AD = CE = 100 \text{ yds.} - 17{\cdot}32 \text{ yds.} =$ $82{\cdot}68$ yds.

V.

1. Radius of equator $= 4000$ m.; $\therefore$ required angle in circ. meas. $=$ $\frac{1309}{4000} = \frac{31416}{4000 \times 24} = \frac{5}{48}\pi$, and in degrees, &c. $= \frac{1309}{4000} \cdot \frac{180°}{\pi} =$ $\frac{1309}{4000} \cdot \frac{180°}{3{\cdot}1416} = \frac{1800°}{4 \times 24} = 18°\,45'$.

2. $$f = \frac{1 - \tan^2 A \tan^2 B}{\tan^2 A \tan^2 B} \cdot \frac{\cos^2 A \cos^2 B}{\cos^2 A \cos^2 B} = \frac{\cos^2 A \cos^2 B - \sin^2 A \sin^2 B}{\sin^2 A \sin^2 B}$$
$$= \frac{\cos^2 A\,(1 - \sin^2 B) - (1 - \cos^2 A)\sin^2 B}{\sin^2 A \sin^2 B} = l.$$

3. Squaring (1), $\sin^2\alpha + 2\sin\alpha\sin\beta + \sin^2\beta = 2$; from (2), $4\sin\alpha\sin\beta = 2$; subtracting, $(\sin\alpha - \sin\beta)^2 = 0$, $\therefore \sin\alpha = \sin\beta$; and from (2), $\sin^2\alpha = \frac{1}{2}$, $\therefore \sin\alpha = \frac{1}{\sqrt{2}}$, and $\alpha = \frac{1}{4}\pi = \beta$.

4. $\tan^2\theta + 1 = 2\tan\theta$, $\therefore (\tan\theta - 1)^2 = 0$, $\therefore \tan\theta = 1$, $\therefore \theta = \frac{1}{4}\pi$; $\therefore \sin\theta + \cos\theta = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$.

5. Remembering that $\frac{1}{2}\theta$ varies from 0 to π, while θ varies from 0 to 2π; we have simply to trace $2 \operatorname{cosec} \frac{1}{2}\theta$ through the first two quadrants, in which it is +ve. Now, in fig. of Art. 45, $\operatorname{cosec} A = \frac{r}{y}$, and r does not vary.

When $\frac{1}{2}\theta$ is	0	between 0 and $\frac{1}{2}\pi$	$\frac{1}{2}\pi$	between $\frac{1}{2}\pi$ and π	π
y is	0	increasing	r	decreasing	0
$\therefore 2 \operatorname{cosec} \frac{1}{2}\theta$ is	∞	decreasing	2	increasing	∞

6. In fig. (31), take C, D, H as Calais, Dover, and Hastings; then $CD = 13\sqrt{3}$ m., $DH = 39$ m., $\angle CDH = 56\frac{1}{4}° + 33\frac{3}{4}° = 90°$, $\therefore \tan H = \frac{CD}{DH} = \frac{13\sqrt{3}}{39} = \frac{1}{\sqrt{3}}$, $\therefore H = 30°$, $\therefore CHE = 56\frac{1}{4}° - 30° = 26\frac{1}{4}°$.

VI.

1. Let $(n-x)°$ and $x°$ be required parts; then $60\,(n-x) = \frac{9}{10}x \times 100$; $\therefore 54\,(n-x) = 100x$, $\therefore x = \frac{54}{154}n = \frac{27}{77}n$, and $n - x = \frac{50}{77}n$.

2. $\because \sec^2\theta - \tan^2\theta = 1$, $\therefore (\sec^2\theta - \tan^2\theta)^2 = 1$; squaring, &c., $f = l$.

3. $\dfrac{\cos^2 A}{\sin^2 C} + \dfrac{\cos^2 B}{\sin^2 C} = \cos^2\theta + \cos^2(90^\circ - \theta) = \cos^2\theta + \sin^2\theta = 1,$

$\therefore\ 1 - \sin^2 A + 1 - \sin^2 B = \sin^2 C, \quad \therefore\ f = l.$

4. In fig. to Art. 45, $\tan A = \dfrac{PN}{ON}$, $\sin A = \dfrac{PN}{OP}$. Now, as A varies, ON is always less than OP in the right-angled triangle PON, unless they coincide, which happens when $A = n \,.\, 180^\circ$, $\therefore\ \tan A > \sin A$, unless $A = n \,.\, 180^\circ$.

In same fig., $\sec A = \dfrac{OP}{ON}$, $\tan A = \dfrac{PN}{ON}$, and OP is always $> PN$, except when they coincide, which happens when $A = n \,.\, 180^\circ + 90^\circ$, $\therefore\ \sec A > \tan A$, except $A = n \,.\, 180^\circ + 90^\circ$.

5. $\sin 150^\circ = \sin(180^\circ - 150^\circ) = \sin 30^\circ = \frac{1}{2}$.
$\sec 585^\circ = \sec(585^\circ - 360^\circ) = \sec 225^\circ = \sec(180^\circ + 45^\circ)$
$= -\sec 45^\circ = -\sqrt{2}$.
$\cot \frac{23}{6}\pi = \cot(4\pi - \frac{1}{6}\pi) = \cot(-\frac{1}{6}\pi) = -\cot\frac{1}{6}\pi = -\sqrt{3}$.

6. Taking fig. (32), OCA is a right-angled triangle (Euc. III. 18); and $\therefore\ BAC = 60^\circ$ and $BA = BC$, $\therefore\ ABC$ is equilateral;

$\therefore\ OC = CA \tan 30^\circ = t\dfrac{1}{\sqrt{3}}$, and $BC = CA = t$.

VII.

1. $f = \dfrac{\tan A - \sin A}{\tan A}\,\dfrac{\cos A}{\cos A} = \dfrac{\sin A - \sin A \cos A}{\sin A} = 1 - \cos A$, and

$l = \dfrac{\operatorname{cosec} A - \cot A}{\operatorname{cosec} A}\,\dfrac{\sin A}{\sin A} = 1 - \cos A; \quad \therefore\ f = l.$

2. $f = \dfrac{\sin^3\theta}{\cos\theta} + \dfrac{\cos^3\theta}{\sin\theta} + 2\sin\theta\cos\theta = \dfrac{\sin^4\theta + 2\sin^2\theta\cos^2\theta + \cos^4\theta}{\cos\theta\sin\theta}$

$= \dfrac{(\sin^2\theta + \cos^2\theta)^2}{\cos\theta\sin\theta} = \dfrac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} = \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = l.$

3. By hypothesis, $\dfrac{1}{n^2\sin^2 A} - \dfrac{\tan^2 B}{\sin^2 A} = \sec^2\phi - \tan^2\phi = 1.$

$\therefore\ \dfrac{\tan^2 B}{\sin^2 A} = \dfrac{1}{n^2\sin^2 A} - 1; \quad \therefore\ n^2\tan^2 B = 1 - n^2\sin^2 A$, or

$n^2(\sec^2 B - 1) = 1 - n^2(1 - \cos^2 A), \quad \therefore\ n^2\sec^2 B = 1 + n^2\cos^2 A,$
$\therefore\ \cos^2 B =$ &c.

4. $\cdot 5 = \sin 30^\circ = \sin 150^\circ$; adding $+$ve and $-$ve revolutions, by Art. 58, we get from 150°, -210°, -570°, 510°; and from 30°, -330°, -690°, 390°, 750°.

5. Let (fig. 33) $AOB = 2A$, AB being an arc of centre O; draw BC perp. to OA, and OD bisecting AOB. Then, $AB > BC$, $\because\ ACB = 90^\circ$ and is therefore $> BAC$, $\therefore\ 2AD > BC$, $\therefore\ 2\dfrac{AD}{OA} > \dfrac{BC}{OA}$, *i.e.* $> \dfrac{BC}{OB}$,

6. $\tan(90^\circ + A) = -\cot A$ (Art. 60), $\therefore f = 1 - \tan A \cot A = 1 - 1 = 0.$

7. $f = (\operatorname{cosec}^2 A - 1) - (\sec^2 A - 1) = \dfrac{1}{\sin^2 A} - \dfrac{1}{\cos^2 A} = \dfrac{\cos^2 A - \sin^2 A}{\sin^2 A \cos^2 A}$
$= \dfrac{4 \cos 2A}{\sin^2 2A} = l.$

8. $f = \cos A \cos 2A + \sin A \sin 2A - (\sin A \cos 2A + \cos A \sin 2A)$
$= \cos (2A - A) - \sin (A + 2A) = l.$

VIII.

1. $\cdot 1309^c + 2° \, 24' = \frac{1}{24}(3 \cdot 1416^c) + 2\frac{2}{5}° = \frac{1}{24}\pi^c + \frac{12}{5}° = \frac{200^g}{24} + \frac{12^g}{5} \times \frac{10}{9}$
$= \frac{25^g}{3} + \frac{8^g}{3} = \frac{33^g}{3} = 11.$

2. $f = \dfrac{\cos\theta}{\sin\theta} - \dfrac{\cos^2\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta} = \dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta} - \dfrac{1 - \sin^2\theta}{\sin\theta}$
$= \dfrac{\cos^2\theta + \sin^2\theta}{\sin\theta \cos\theta} - \dfrac{1}{\sin\theta} + \sin\theta = \operatorname{cosec}\theta \sec\theta - \operatorname{cosec}\theta + \sin\theta = l.$

3. $1 = \operatorname{cosec}^2\theta - \cot^2\theta = \dfrac{\sin^2 B}{\cos^2 A} - \dfrac{n^2}{m^2 \cos^2 A}$; $\therefore \; m^2 \cos^2 A = m^2 \sin^2 B - n^2$,
$\therefore \; m^2 \sin^2 B = m^2 \cos^2 A + n^2, \quad \therefore \; f = l.$

4. $\operatorname{cosec}\theta = \dfrac{2 + \sqrt{3} \pm \sqrt{(7 + 4\sqrt{3} - 8\sqrt{3})}}{2\sqrt{3}} = \dfrac{2 + \sqrt{3} \pm (2 - \sqrt{3})}{2\sqrt{3}} = \dfrac{4}{2\sqrt{3}}$
or $\dfrac{2\sqrt{3}}{2\sqrt{3}} = \dfrac{2}{\sqrt{3}}$ or 1; $\therefore \; \theta = \frac{1}{3}\pi$ or $\frac{2}{3}\pi$ or $\frac{1}{2}\pi$.

5. $\dfrac{2n\sqrt{(1 - n^2)}}{1 - 2n^2} = \dfrac{\sin 2A}{\cos 2A} = \dfrac{2 \sin A \cos A}{1 - 2\sin^2 A} = \dfrac{2 \sin A \sqrt{(1 - \sin^2 A)}}{1 - 2\sin^2 A}$;
$\therefore \; \sin A = n$, by inspection.

6. $l = \dfrac{\sin(\frac{1}{4}\pi + \theta)\cos(\frac{1}{4}\pi - \theta) - \cos(\frac{1}{4}\pi + \theta)\sin(\frac{1}{4}\pi - \theta)}{\sin(\frac{1}{4}\pi + \theta)\cos(\frac{1}{4}\pi - \theta) + \cos(\frac{1}{4}\pi + \theta)\sin(\frac{1}{4}\pi - \theta)}$
$= \dfrac{\sin(\frac{1}{4}\pi + \theta - \frac{1}{4}\pi + \theta)}{\sin(\frac{1}{4}\pi + \theta + \frac{1}{4}\pi - \theta)} = \dfrac{\sin 2\theta}{\sin \frac{1}{2}\pi} = f.$

7. $2f = \sin 2A - \sin 2B + \sin 2B - \sin 2C + \sin 2C - \sin 2A = 0.$

8. $f = 2 \sin (A + B) \cos 2B \sec 2B = 2 \sin (A + B) =$
$2 \sin (A + B) \sin (A - B) \operatorname{cosec}(A - B) = (\cos 2B - \cos 2A) \operatorname{cosec}(A - B).$

IX.

1. $\cos^2 A + 2 \cos A \cos B + \cos^2 B = \frac{1}{4}(4 + 2\sqrt{3})$, and $4 \cos A \cos B =$
$\frac{1}{4}(4\sqrt{3})$; $\therefore \; \cos^2 A - 2\cos A \cos B + \cos^2 B = \frac{1}{4}(4 - 2\sqrt{3})$;
$\therefore \; \cos A - \cos B = \frac{1}{2}(1 - \sqrt{3})$. Adding, &c., $\cos A + \cos B = \frac{1}{2}(1 + \sqrt{3})$,
$2 \cos A = 1$, $2 \cos B = \sqrt{3}$, $\therefore \; A = 60°$, $B = 30°$.

2. $f = \sin\theta + \dfrac{\sin^2\theta}{\cos\theta} + \cos\theta + \dfrac{\cos^2\theta}{\sin\theta} - \sec\theta$
$= \sin\theta + \dfrac{\cos^2\theta}{\sin\theta} + \dfrac{\sin^2\theta}{\cos\theta} + \cos\theta - \sec\theta$
$= \dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta} + \dfrac{\sin^2\theta + \cos^2\theta}{\cos\theta} - \sec\theta = \operatorname{cosec}\theta + \sec\theta - \sec\theta = \operatorname{cosec}\theta$
$= \sqrt{(\cot^2\theta + 1)} = \sqrt{\left(\dfrac{a^2 - b^2}{b^2} + 1\right)} = \sqrt{\dfrac{a^2}{b^2}} = l.$

4. $L\cot 35^\circ 20' = 10 + \log\dfrac{1}{\tan 35^\circ 20'} = 10 - \log\tan 35^\circ 20'$
$= 10 - (L\tan 35^\circ 20' - 10) = 20 - 9{\cdot}850593 = 10{\cdot}149407.$

5. $f = \cos 9^\circ + \sin(90 + 9^\circ) - (\sin 39^\circ + \sin 21^\circ)$
$= \cos 9^\circ + \cos 9^\circ - 2\sin 30^\circ \cos 9^\circ$
$= \cos 9^\circ = \cos(9^g \times \frac{10}{9}) = \cos 10^g = \sin 90^g.$

6. Let x = reqd. base; then, if n be any logarithm to base 10,
$10^n = x^{2n}$, $\therefore\ x^2 = 10$, $\therefore\ x = \sqrt{10} = 3{\cdot}162.$

7. $9{\cdot}68472 = L\tan x$ $\qquad \frac{7}{32}$ of $60'' = \frac{105}{8}'' = 13''$,
$9{\cdot}68465 = L\tan 25^\circ 49'$ $\qquad x = 25^\circ 49' 13''$.
$\overline{7}$

8. $\frac{1}{3}(\log 2 + \log\sec A) = -\log 5 + \frac{1}{3}(\log\cos^2 B - \log\sin C)$,
$\therefore\ 3\log 2 + 3(L\sec A - 10) = -6\log 5 + 2[2(L\cos B - 10) - (L\sin C - 10)]$,
$\therefore\ 3\log 2 + 3L\sec A = 4L\cos B - 2L\sin C + 10 - 6\log 5.$

XIII.

1. While A is between 0 and 180°, $3A$ must be between 0 and 540°; and $\sin 3A = \frac{1}{2} = \sin 30^\circ = \sin 150^\circ$; also $\sin 30^\circ = \sin 390^\circ$, and $\sin 150^\circ = \sin 510^\circ$, $\therefore\ A = 10^\circ, 50^\circ, 130^\circ$, or 170°.

2. (1) $f = \sin(\frac{1}{2}\pi - P) + \sin Q = 2\sin\frac{1}{2}(\frac{1}{2}\pi - P + Q)\cos\frac{1}{2}(\frac{1}{2}\pi - P - Q) = l$,
(2) $f = \sin(\frac{1}{2}\pi - P) - \sin Q = 2\cos\frac{1}{2}(\frac{1}{2}\pi - P + Q)\sin\frac{1}{2}(\frac{1}{2}\pi - P - Q) = l$.

3. $\qquad -1{\cdot}8753145 = \bar{2}{\cdot}1246855 \qquad {\cdot}1246998$
$\qquad\qquad {\cdot}1246672 \qquad {\cdot}1246672$
$\qquad\qquad \overline{183} \qquad \overline{326}$

$\frac{183}{326} = \frac{5}{9}$ nearly $= {\cdot}556$ nearly, $\therefore$ reqd. no. $= {\cdot}013325556$.

4. $l = 2\cos A\sin B\sin(A+B) + 2\sin A\cos B\sin(A+B)$
$\quad - 2\sin A\sin B\cos(A+B)$
$= \sin A[\sin(A+B)\cos B - \cos(A+B)\sin B]$
$\quad + \sin B[\sin(A+B)\cos A - \cos(A+B)\sin A]$
$\quad + \sin(A+B)(\sin A\cos B + \cos A\sin B)$
$= \sin A.\sin A + \sin B.\sin B + \sin(A+B).\sin(A+B) = f.$

5. Diff. for $60'' = {\cdot}61972 - {\cdot}61936 = 36$; $\frac{52}{60}$ of $36 = 31$,
$\therefore\ L\cot 67^\circ 23' 52'' = 9{\cdot}61972 - 31 = 9{\cdot}61941$,
$\therefore\ \log\cot 67^\circ 23' 52'' = \bar{1}{\cdot}61941.$

6. $$l = \tfrac{1}{4}\left\{\left(\frac{\sin B}{\sin A} + \frac{\sin C}{\sin A} - \frac{\sin A}{\sin A}\right)\left(\frac{\sin C}{\sin B} + \frac{\sin A}{\sin B} - \frac{\sin B}{\sin B}\right)\right\}$$
$$= \tfrac{1}{4}\left\{\left(\frac{b}{a} + \frac{c}{a} - \frac{a}{a}\right)\left(\frac{c}{b} + \frac{a}{b} - \frac{b}{b}\right)\right\} = \tfrac{1}{4}\left(\frac{b+c-a}{a}\cdot\frac{c+a-b}{b}\right)$$
$$= \frac{2(s-a).2(s-b)}{4ab} = \frac{(s-a)(s-b)}{ab} = f.$$

7. ${\cdot}78329 \qquad {\cdot}78329$
${\cdot}78313 \qquad {\cdot}78321 \qquad \frac{8}{16}$ of $60'' = 30''$, $\therefore\ x = 52^\circ 37' 30''$.
$\overline{16} \qquad \overline{8}$

8. Suppose the $\angle$s are A and B, then a, b are the sides,

$b = a\dfrac{\sin B}{\sin A}$, $\quad\therefore\ \log b = \log a + L\sin B - L\sin A,$

$= \log 79{\cdot}063$

$\therefore\ b = 79{\cdot}063.$

$$\begin{array}{r} 1{\cdot}7403627 \\ \underline{9{\cdot}9764927} \\ 11{\cdot}7168554 \\ \underline{9{\cdot}8188779} \\ 1{\cdot}8979775 \end{array}$$

XIV.

1. Expression $= 1 + \cos 2\theta + \cos^2 2\theta - (1 + \cos 2\theta)\cos 2\theta = 1.$

2. In fig. to Art. 58, suppose $PON = 45° - A$, then $OPN = 90° - (45° - A) = 45° + A$;

$\therefore\ \tan(45° - A)\tan(45° + A) = \dfrac{PN}{ON}\cdot\dfrac{ON}{PN} = 1.$

3. $2f = 2\sin^2\frac{1}{2}A - 2\sin^2\frac{1}{2}B = 1 - \cos A - (1 - \cos B) = \cos B - \cos A$

$= \dfrac{a}{c} - \dfrac{b}{c} = 2l.$

4. $A + B > C$, $\therefore\ \sin\frac{1}{2}(A + B) > \sin\frac{1}{2}C$, or $\sin\frac{1}{2}A\cos\frac{1}{2}B + \cos\frac{1}{2}A\sin\frac{1}{2}B > \sin\frac{1}{2}C$; but $\cos\frac{1}{2}B$, $\cos\frac{1}{2}A$ are proper fractions, $\therefore$ *à fortiori*, $\sin\frac{1}{2}A + \sin\frac{1}{2}B > \sin\frac{1}{2}C$.

5. $f = \cos^2\alpha + \cos^2\beta - [\cos(\alpha + \beta) + \cos(\alpha - \beta)]\cos\omega$

$= \cos^2\alpha + \cos^2\beta - \cos(\alpha - \beta)\cos(\alpha + \beta) - \cos^2\omega$

$= \cos^2\alpha + \cos^2\beta - (\cos^2\alpha - \sin^2\beta) - \cos^2\omega$

$= 1 - \cos^2\omega = l.$

6. $$\begin{array}{r} 3786924 \\ \underline{3786802} \\ 122 \end{array}$$ $\frac{122}{581}$ of $10'' = 2''$; $\therefore$ reqd. $\angle = 67° 18' 32''$.

7. $\cos B = \dfrac{a}{c}$, $\therefore\ c = \dfrac{a}{\cos B}$

$\therefore \log c = \log a - (L\cos B - 10)$

$= 2{\cdot}05176,$

$\therefore\ c = 112{\cdot}658.$

$$\begin{array}{r} 1{\cdot}83518 \\ \underline{\bar{1}{\cdot}78342} \\ 2{\cdot}05176 \\ \underline{{\cdot}05154} \\ 22 \end{array}$$

$\frac{22}{38} = {\cdot}58$

8. $\sin A = \dfrac{a}{c}\sin C = \dfrac{15}{35}\cdot\dfrac{\sqrt{3}}{2}$

$= \dfrac{3\sqrt{3}}{14} = \dfrac{3^{\frac{3}{2}}}{14}$

$\therefore\ L\sin A = \frac{3}{2}\log 3 - \log 14 + 10$

$= 10{\cdot}715681 - 1{\cdot}146128 = 9{\cdot}569553.$

$$\begin{array}{rr} 569804 & 569553 \\ \underline{569488} & \underline{569488} \\ 316 & 65 \end{array}$$

$\frac{65}{316}$ of $60'' = 13''$ nearly,

$A = 21° 47' 13''.$

XV.

1. $\sin\theta - \operatorname{cosec}\theta = \dfrac{\sin^2\theta - 1}{\sin\theta} = -\dfrac{\cos^2\theta}{\sin\theta}$, $\cos\theta - \sec\theta = -\dfrac{\sin^2\theta}{\cos\theta}$;

$\therefore\ f = 4\dfrac{\cos^4\theta}{\sin^2\theta}\cdot\dfrac{\sin^4\theta}{\cos^2\theta} + (1 + \sin 2\theta)(1 - \sin 2\theta)$

$= \sin^2 2\theta + 1 - \sin^2 2\theta = 1.$

2. $f = (\sin^2\theta + \cos^2\theta)(\sin^4\theta - \sin^2\theta\cos^2\theta + \cos^4\theta) + 3\sin^2\theta\cos^2\theta$
$= \sin^4\theta + 2\sin^2\theta\cos^2\theta + \cos^4\theta = (\sin^2\theta + \cos^2\theta)^2 = 1.$

3.

$$\begin{array}{r} 9359018 \\ 9358894 \\ \hline 124 \end{array} \qquad \begin{array}{r} 9358921 \\ 9358894 \\ \hline 27 \end{array} \qquad \begin{array}{l} \frac{27}{124} \text{ of } 10'' = 2''\cdot18, \\ A = 59^\circ\, 37'\, 42''\cdot18. \end{array}$$

4. $f = 1 - \tan^2\alpha\tan^2\beta + \dfrac{\tan^2\alpha - \tan^2\beta}{\sin(\alpha+\beta)\sin(\alpha-\beta)}$

$= 1 - (\sec^2\alpha - 1)(\sec^2\beta - 1) + \dfrac{\tan^2\alpha - \tan^2\beta}{\sin^2\alpha - \sin^2\beta}$

$= 1 - \sec^2\alpha\sec^2\beta + \sec^2\alpha + \sec^2\beta - 1 + \dfrac{1}{\cos^2\alpha\cos^2\beta} = l.$

5. $l = 2\sin\frac{1}{2}(\alpha+\beta)\left[\cos\frac{1}{2}(\alpha-\beta) - \cos\frac{1}{2}(\alpha+\beta+2\gamma)\right]$
$= \sin\alpha + \sin\beta - [\sin(\alpha+\beta+\gamma) - \sin\gamma] = f.$

6. $\dfrac{b+c}{a} = \dfrac{b}{a} + \dfrac{c}{a} = \cot A + \operatorname{cosec} A = \dfrac{\cos A + 1}{\sin A} = \dfrac{2\cos^2\frac{1}{2}A}{2\sin\frac{1}{2}A\cos\frac{1}{2}A}$

$= \dfrac{\cos\frac{1}{2}A}{\sin\frac{1}{2}A}, \quad \therefore f = l.$

7. $a^2 = c^2 - b^2 = (c+b)(c-b) = 89168 \times 23686,$
$\therefore \log a = \frac{1}{2}(\log 89168 + \log 23686) = 4\cdot66235, \quad \therefore a = 45956.$

8. Complete the parallelogram AD on AB, AC; join AD meeting BC in E, AE is bisector. Now $ACD = 180^\circ - A$, and $AD = 2AE$, $\therefore (2AE)^2 = AC^2 + CD^2 - 2AC \,.\, CD\cos ACD = b^2 + c^2 + 2bc\cos A$, $\therefore f = l.$

9. Suppose O' (fig. 32) centre of inscribed circle and ρ its radius,

$\rho = O'A\sin A = (OA - r - \rho)\sin A = \left(\dfrac{r}{\sin A} - r - \rho\right)\sin A,$

$\therefore \rho(1 + \sin A) = r(1 - \sin A), \quad \therefore \rho = \dfrac{1 - \sin A}{1 + \sin A}r.$

10. $s = 900, \; s - a = 400, \; s - b = 300, \; s - c = 200,$
$\Delta^2 = 900 \times 400 \times 300 \times 200 = 3^3 \,.\, 2^3 \,.\, 10^8,$
$2\log\Delta = 3\log 3 + 3\log 2 + 8 = 10\cdot33445, \quad \therefore \log\Delta = 5\cdot167225,$
diff. for $1 = 30$, pro. pt. $= 20\cdot5$, $\dfrac{20\cdot5}{30} = \cdot68\dot{3},$

$\therefore \Delta = 146968\cdot\dot{3}$ sq. links $= 1$ ac. 4 sq. ch. $6968\cdot\dot{3}$ sq. links.

XVI.

1. For reqd. circ. me., $180^\circ : \dfrac{858^\circ}{100 \times 60 \times 60} = \pi : x = \dfrac{3\cdot1416 \times 858}{100 \times 60 \times 60 \times 180}$
$= \dfrac{1309 \times 143}{4500000000} = \cdot0000416$ nearly. If $x =$ reqd. dist. of sun in m.,

$\dfrac{4000}{x} = \cdot0000416, \quad \therefore x = \dfrac{4000}{\cdot0000416} = 96{,}000{,}000$ about.

2. $\log\left(\sqrt[14]{\tfrac{1}{3}}\right)^{187} = \log\left(\tfrac{1}{3}\right)^{\frac{187}{14}} = \frac{187}{14}(-\log 3) = -6\cdot37297736 = \bar{7}\cdot6270226.$
$= \log\cdot00000042366.$

3. $\because A+B=180^\circ-C=90^\circ$, $\therefore \frac{1}{2}(A+B)=45^\circ$, $\therefore \cot\frac{1}{2}(A+B)=1$,
$\therefore \dfrac{\cot\frac{1}{2}A\cot\frac{1}{2}B-1}{\cot\frac{1}{2}A+\cot\frac{1}{2}B}=1$, $\therefore f=l$.

4. $4f=2(\cos 40^\circ-\cos 60^\circ)\cos 20^\circ=\cos 60^\circ+\cos 20^\circ-2\times\frac{1}{2}\cos 20^\circ=\frac{1}{2}$.

5. $2\log\tan A=\log\cdot 96386=\bar{1}\cdot 9840140$, $\therefore L\tan A=9\cdot 9920070$;
$\therefore$ pro. part $=20070-19143=927$; and $\frac{927}{2527}$ of $60''=22''$,
$\therefore A=44^\circ\,28'\,22''$.

6. $a\left(\dfrac{\sin A}{\cos A}-\dfrac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A+B)}\right)=b\left(\dfrac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A+B)}-\dfrac{\sin B}{\cos B}\right)$
$\therefore a\left[\sin A\cos\frac{1}{2}(A+B)-\cos A\sin\frac{1}{2}(A+B)\right]\cos B$
$=b\left[\sin\frac{1}{2}(A+B)\cos B-\cos\frac{1}{2}(A+B)\sin B\right]\cos A$,
$\therefore a\sin\frac{1}{2}(A-B)\cos B=b\sin\frac{1}{2}(A-B)\cos A$, $\therefore f=l$.

7. Expression $=\dfrac{1}{\cos A}\sin A+\dfrac{\tan B\cot C+1}{\cot C}=\tan A+\tan B+\tan C$,
which is symmetrical with respect to A, B, and C.

8. $\angle OAE=90^\circ-C$, $\angle OBD=90^\circ-C$, $\therefore OA=\dfrac{OE}{\sin OAE}=\dfrac{OE}{\cos C}$.
and $OD=OB\sin OBD=BO\cos C$, $\therefore AO\,.\,OD=BO\,.\,OE$;
and similarly $=CO\,.\,OF$.

9. Let $C=120^\circ$, and $b=4a$. Then $\sin B=\dfrac{b}{a}\sin A=4\sin(120^\circ+B)$
$=4\left(\frac{1}{2}\sqrt{3}\cos B-\frac{1}{2}\sin B\right)$, $\therefore \cot B=\frac{1}{2}\sqrt{3}$. Similarly $\cot A=3\sqrt{3}$.

10. Constructing as in fig. 34, let $DE=x$, $CD=h$, $DAE=DBE=15^\circ$;
then a circle will go round $ABED$; $\therefore h(h+x)=ab$;

$\therefore \tan DAE=\tan(CAE-CAD)=\dfrac{\frac{h+x}{a}-\frac{h}{a}}{1+\frac{h(h+x)}{a^2}}=\dfrac{\frac{x}{a}}{1+\frac{b}{a}}$,
$\therefore x=(a+b)\tan 15^\circ=$ &c.

XVII.

1. Let $\theta=$ circ. meas. reqd.; then arc $=r\theta$, $\therefore \dfrac{2r\theta}{r\theta+2r}=\dfrac{2\theta}{\theta+2}=3-\sqrt{5}$,
$\therefore \theta(\sqrt{5}-1)=6-2\sqrt{5}=(\sqrt{5}-1)^2$, $\therefore \theta=\sqrt{5}-1$.

2. $\tan\theta=\dfrac{\sin\theta}{\sqrt{(1-\sin^2\theta)}}=\dfrac{5}{13}\cdot\dfrac{1}{\sqrt{(1-\frac{25}{169})}}=\dfrac{5}{13}\cdot\dfrac{13}{\sqrt{144}}=\dfrac{5}{12}$;
$\tan 2\theta=2\times\dfrac{5}{12}\times\dfrac{1}{1-\frac{25}{144}}=\dfrac{5}{6}\times\dfrac{144}{119}=\dfrac{120}{119}$.

3. Make $xOP=A$ of any magnitude (fig. 35), and $yOP'=-A$; then $xOP'=90^\circ-A$. Making $OP'=OP$, the triangles yOP and xOP' are equal in all respects, having $PN=ON'$ in magnitude. These also have the same sign; for $\because$ the angles xOP and yOP' are equal in magnitude, when P is above xO, P' is always to the right of yO, and when below, to the left. Therefore in magnitude and sign
$PN=ON'$, $\therefore \dfrac{PN}{OP}=\dfrac{ON'}{OP'}$, $\therefore \sin A=\cos(90^\circ-A)$.

4. By Arith. Prog., $4 \sec (\frac{2}{3}\pi + \theta) = 2 \sec \theta + 2 \sec (\frac{2}{3}\pi - \theta)$,

$\therefore\ 4 \cos \theta \cos (\frac{2}{3}\pi - \theta) = 2 \cos (\frac{2}{3}\pi + \theta) \cos (\frac{2}{3}\pi - \theta) + 2 \cos (\frac{2}{3}\pi + \theta) \cos \theta$,

$\therefore\ 2 \cos \frac{2}{3}\pi + 2 \cos (\frac{2}{3}\pi - 2\theta) = \cos \frac{4}{3}\pi + \cos 2\theta + \cos (\frac{2}{3}\pi + 2\theta) + \cos \frac{2}{3}\pi$,

$\therefore\ 2 \cos (\frac{2}{3}\pi - 2\theta) = 2 \cos (\frac{1}{3}\pi + 2\theta) \cos \frac{1}{3}\pi = \cos (\frac{1}{3}\pi + 2\theta)$

$= -\cos (\frac{2}{3}\pi - 2\theta)$, $\therefore\ \cos (2\theta - \frac{2}{3}\pi) = 0 = \cos (2n\pi \pm \frac{1}{2}\pi)$,

$\therefore\ 2\theta = 2n\pi + \frac{2}{3}\pi \pm \frac{1}{2}\pi$, $\therefore\ \theta = n\pi + \frac{1}{3}\pi \pm \frac{1}{4}\pi$.

5. $$f = \frac{1}{\cos^2 \frac{1}{2}\theta \cos \theta} \cdot \frac{\operatorname{cosec}^2 \frac{1}{2}\theta - \operatorname{cosec}^2 \frac{3}{2}\theta}{\operatorname{cosec}^2 \frac{3}{2}\theta} = \frac{\sin^2 \frac{3}{2}\theta}{\cos^2 \frac{1}{2}\theta \cos \theta} \cdot \frac{\sin^2 \frac{3}{2}\theta - \sin^2 \frac{1}{2}\theta}{\sin^2 \frac{3}{2}\theta \sin^2 \frac{1}{2}\theta}$$

$$= \frac{4 \sin 2\theta \sin \theta}{\sin^2 \theta \cos \theta} = \frac{8 \sin^2 \theta \cos \theta}{\sin^2 \theta \cos \theta} = 8.$$

6. Circumf. $= \pi \,.\, 2r = 3{\cdot}1416$ in. $\times 1\frac{1}{4} = 3{\cdot}1416$ in. $+ {\cdot}7854$ in. $= 3{\cdot}927$ in.

7. $\sin A \sin (B - C) = \sin (A - B) \sin C$, $\therefore\ \sin (B + C) \sin (B - C) =$
$\sin (A + B) \sin (A - B)$, $\therefore\ \sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$;
$\therefore$ proportionally, $b^2 - c^2 = a^2 - b^2$.

8. $\log c = \log a + L \sin C - L \sin A$
$= 2{\cdot}09513$,
$\therefore\ c = 124{\cdot}48$.

2·03933	11·96350
9·92417	9·86837
11·96350	2·09513

9. $$f = \tan^{-1} \frac{\dfrac{a}{b+c} + \dfrac{b}{a+c}}{1 - \dfrac{ab}{(b+c)(a+c)}} = \tan^{-1} \frac{a^2 + b^2 + (a+b)\,c}{(a+b)\,c + c^2}$$

$$= \tan^{-1} \frac{c^2 + (a+b)\,c}{(a+b)\,c + c^2} = \tan^{-1} 1 = l.$$

10. $c^2 = a^2 + b^2 - 2ab \cos C = a^2 + b^2 - 2ab\,(1 - 2 \sin^2 \frac{1}{2}C)$

$$= (a-b)^2 + 4ab \sin^2 \tfrac{1}{2}C = (44)^2 + 4 \,.\, 169 \,.\, 125 \left(\frac{33}{65}\right)^2$$

$$= 1936 + 4 \,.\, (13)^2 \,.\, 5^3 \left(\frac{33}{13 \,.\, 5}\right)^2 = 154^2,\quad c = 154.$$

XVIII.

1. Let a° = reqd. unit, and A° = any $\angle$; then $\dfrac{A}{a} = \angle A^\circ$, measured in terms of reqd. unit. Hence, by question, $\dfrac{A}{a} + A = \frac{10}{9} A$;
$\therefore\ \dfrac{1}{a} = \dfrac{1}{9}$, $\therefore\ a^\circ = 9^\circ$.

2. $f = 2\,(\sin^2 \theta + \cos^2 \theta)(\sin^4 \theta - \sin^2 \theta \cos^2 \theta + \cos^4 \theta) - 3 \sin^4 \theta - 3 \cos^4 \theta + 1$
$= 1 - (\sin^4 \theta + 2 \sin^2 \theta \cos^2 \theta + \cos^4 \theta) = 1 - (\sin^2 \theta + \cos^2 \theta)^2 = 1 - 1 = 0$.

3. $f = \sin^2 A + \cos^2 A + \sin^2 B + \cos^2 B - 2\,(\cos A \cos B + \sin A \sin B)$
$- 2\,[1 - \cos (A - B)] = 1 + 1 - 2 \cos (A - B) - 2 + 2 \cos (A - B) = 0$.

4. Expression $= (-1)^3\,(-\tan \frac{1}{3}\pi)^3\,(-\tan \frac{1}{6}\pi)^5\,[4 \tan^4 \frac{1}{4}\pi + 7\,(-\tan \frac{1}{4}\pi)]$

$$= -(-\sqrt{3})^3 \left(-\frac{1}{\sqrt{3}}\right)^5 (4 - 7) = -\left(-\frac{1}{\sqrt{3}}\right)^2 (-3) = 1.$$

5. $\sin \theta = \pm 1 = \sin (\pm \frac{1}{2}\pi)$, $\therefore\ \theta = n\pi \pm \frac{1}{2}\pi$, which are all included in $n\pi + \frac{1}{2}\pi$.

6. $f = \cos^{-1}(1-a)+\cos^{-1}(1-b)$
$= \cos^{-1}\left\{(1-a)(1-b)-\sqrt{[(2a-a^2)(2b-b^2)]}\right\}$
$= \operatorname{vers}^{-1}\left\{1-(1-a-b+ab)+\sqrt{[ab(2-a)(2-b)]}\right\} = l.$

7. $\dfrac{\sin A+\sin B+\sin C}{2\sin A\sin B\sin C} = \left(\dfrac{2\Delta}{bc}+\dfrac{2\Delta}{ca}+\dfrac{2\Delta}{ab}\right) \div 2\dfrac{2\Delta\,.\,2\Delta\,.\,2\Delta}{bc\,.\,ca\,.\,ab}$
$= \left(\dfrac{1}{bc}+\dfrac{1}{ca}+\dfrac{1}{ab}\right)\times\dfrac{a^2b^2c^2}{8\Delta^2} = \dfrac{2s}{abc}\times\dfrac{a^2b^2c^2}{8\Delta^2} = \dfrac{abc}{4\Delta}\div\dfrac{\Delta}{s} = \dfrac{R}{r}.$

8. $\sin^{-1}\cos(x-\alpha) = \cos^{-1}\sin(x-\beta) = \sin^{-1}\sqrt{[1-\sin^2(x-\beta)]}$
$= \sin^{-1}[\cos(x-\beta)]$, $\therefore\ \cos(x-\alpha) = \pm\cos(x-\beta)$,
$\therefore\ \cos^2(x-\alpha)-\cos^2(x-\beta) = 0$, $\therefore\ \sin(\alpha-\beta)\sin(2x-\alpha-\beta) = 0$,
$\therefore\ \sin(2x-\alpha-\beta) = 0 = \sin n\pi$, $\therefore\ x = \frac{1}{2}(n\pi+\alpha+\beta)$.

9. In fig. 2, Art. 73, suppose CD drawn bisecting BB' and $\therefore$ perpendicular to it. Then $\because A=45°$, $CD=AD=\frac{1}{2}(c+c')$, $BD=\frac{1}{2}(c-c')$,
$\therefore\ \cos BCB' = \cos^2 BCD\,(1-\tan^2 BCD) = \dfrac{1-\tan^2 BCD}{1+\tan^2 BCD}$
$= \left\{1-\left(\dfrac{c-c'}{c+c'}\right)^2\right\} \div \left\{1+\left(\dfrac{c-c'}{c+c'}\right)^2\right\} = \dfrac{4cc'}{(c+c')^2}\times\dfrac{(c+c')^2}{2(c^2+c'^2)} = \dfrac{2cc'}{c^2+c'^2}.$

10. Suppose the perpendicular from A upon BC (fig. to Art. 20). Then $AD\times a = 2\Delta$, also H. M. between $\dfrac{\Delta}{s-b}$ and $\dfrac{\Delta}{s-c}$
$= 2\dfrac{\Delta}{s-b}\cdot\dfrac{\Delta}{s-c}\div\left(\dfrac{\Delta}{s-b}+\dfrac{\Delta}{s-c}\right) = \dfrac{2\Delta}{a} = AD.$

XIX.

1. $\sec^2\theta-\sec\theta-2=0$, $\therefore\ \sec\theta=\frac{1}{2}[1\pm\sqrt{(1+8)}]=2$ or -1, $\therefore\ \theta=$&c.
2. $f = 4\cos\alpha\,(\cos^2\alpha-\sin^2\frac{1}{3}\pi) = 4\cos^3\alpha-4\cos\alpha\,\frac{3}{4} = l.$
3. $3\sin 2\theta+2\cos 2\theta = 2+1+\cos 2\theta$, $\therefore\ \cos 2\theta = 3(1-\sin 2\theta)$,
$\therefore\ \cos^2\theta-\sin^2\theta = 3(\cos\theta-\sin\theta)^2$,
$\therefore$ (1) $\cos\theta-\sin\theta = 0$, $\therefore\ \tan\theta = 1 = \tan\frac{1}{4}\pi$, $\therefore\ \theta = n\pi+\frac{1}{4}\pi$;
(2) $\cos\theta+\sin\theta = 3(\cos\theta-\sin\theta)$, $\therefore\ 4\sin\theta = 2\cos\theta$, $\therefore\ \tan\theta = \frac{1}{2}$.
4. $f = 2\sin 54°\cos 15°-2\sin 36°\cos 15° = 2\cos 15°\,(\sin 54°-\sin 36°)$
$= 2\dfrac{\sqrt{3}+1}{2\sqrt{2}}\,.\,2\cos 45°\sin 9° = l.$
5. $2\sin 3A\cos 4A = 0$, $\therefore\ \sin 7A-\sin A = 0$, $\therefore\ \sin 7A = \sin[n\,.\,180°+(-1)^n A]$, $\therefore\ [7-(-1)^n]A = n\,.\,180°$, $\therefore\ A=$Ans.
6. $n\,.\,180°+(-1)^n\,34°$. Let n be even and $= 2m$; the expression becomes $2m\,.\,180°+34°$, and represents angles having the same tangent as $34°$, for they are bounded by the same lines as $34°$. Let n be odd and $= 2m+1$; then the expression is $2m\,.\,180°+180-34°$, which represents angles in the second quadrant, which therefore have *not* the same tangent.
7. $2\pi r = 2m\,.\,407$ yds. $= 3927$ yds.,
$\therefore\ r = \dfrac{3927}{2\times 3{\cdot}1416}$ yds. $= \dfrac{39270000}{62832}$ yds. $= 625$ yds.

8. $\sin(\pi\cos\theta) = \sin(2n\pi \cdot \frac{1}{2}\pi)$, $\therefore\ \pi\cos\theta = (2n+\frac{1}{2})\pi$,
$\therefore\ \cos\theta = 2n+\frac{1}{2} = \pm\frac{1}{2}$ (only possible values) $= \cos\frac{1}{3}\pi$ or $\cos(\pi+\frac{1}{3}\pi)$,
$\therefore\ \theta = 2n\pi\pm\frac{1}{3}\pi$ or $2n\pi\pm(\pi+\frac{1}{3}\pi) = n\pi\pm\frac{1}{3}\pi$.

9. Expression $= \tan^{-1}\dfrac{x\cos\theta}{1-x\sin\theta} - \tan^{-1}\dfrac{x-\sin\theta}{\cos\theta}$

$$= \tan^{-1}\frac{\dfrac{x\cos\theta}{1-x\sin\theta}-\dfrac{x-\sin\theta}{\cos\theta}}{1+\dfrac{x\cos\theta\,(x-\sin\theta)}{\cos\theta\,(1-x\sin\theta)}} = \tan^{-1}\frac{\sin\theta(1-2x\sin\theta+x^2)}{\cos\theta(1-2x\sin\theta+x^2)}$$
$$= \tan^{-1}\tan\theta = \theta.$$

10. $\dfrac{1-\sin 2\theta}{1+\sin 2\theta} = \dfrac{(1-\sin\theta)^2}{1-\sin^2\theta} = \dfrac{1-\sin\theta}{1+\sin\theta}$, $\therefore\ \sin 2\theta = \sin\theta$,

$\therefore\ 2\theta = n\pi + (-1)^n\theta$, $\therefore\ \theta = \dfrac{n}{2-(-1)^n}\pi$.

Equating the cancelled factor to 0, $\sin\theta = 1$, $\therefore\ \theta = 2n\pi+\frac{1}{2}\pi$.

11. Let x ft. = reqd. height; then $2-\sqrt{3} = \tan 15° = \left(\dfrac{x+10}{50}-\dfrac{x}{50}\right)$
$\div\left(1+\dfrac{x(x+10)}{50 \cdot 50}\right) = \dfrac{10}{50} \div \dfrac{2500+x^2+10x}{2500} = \dfrac{500}{2500+x^2+10x}$;
solving, x = imaginary quantity. The impossibility of the problem also follows from $\tan 15°$ being $>\frac{10}{50}$.

12. Let r be radius, then $\alpha r = 2r+\beta r$, $\therefore\ \alpha-\beta = 2$.

XX.

1. $\tan\theta(\tan\theta+\sqrt{3}) = 0$, $\therefore$ (1) $\tan\theta = 0$, $\therefore\ \theta = n\pi$;
(2) $\tan\theta = -\sqrt{3}$, $\therefore\ \theta = n\pi+\frac{2}{3}\pi$.

2. In fig. 36, $AE \cdot CD \cdot BF = c\cos A \cdot b\cos C \cdot a\cos B$ [$= AF \cdot BD \cdot CE$, by symmetry; otherwise] $= b\cos A \cdot c\cos B \cdot a\cos C = AF \cdot BD \cdot CE$.

3. $f = 2\sin 3A(\cos 2A - \cos 12A) = \sin 5A + \sin A - (\sin 15A - \sin 9A = l$.

4. $\tan^2\theta - 2\tan\theta + 1 = \tan\theta + 4\sqrt{(\tan\theta)} + 4$, $\therefore\ \tan\theta - 1 = \sqrt{(\tan\theta)} + 2$;
$\therefore\ \tan\theta - \sqrt{(\tan\theta)} - 3 = 0$, $\therefore\ \sqrt{(\tan\theta)} = \frac{1}{2}(1\pm\sqrt{13})$,
$\therefore\ \tan\theta = \frac{1}{4}(14\pm2\sqrt{13}) = \frac{1}{2}(7\pm\sqrt{13})$.

5. Let $\alpha-2\beta$, $\alpha-\beta$, α, $\alpha+\beta$, $\alpha+2\beta$ be the angles in A. P. Then
$\cos(\alpha-2\beta)+\cos(\alpha-\beta)+\cos(\alpha+\beta)+\cos(\alpha+2\beta) = 0$,
$\therefore\ 2\cos\alpha\cos 2\beta + 2\cos\alpha\cos\beta = 0$, $\therefore\ \cos 2\beta+\cos\beta = 0$,
$\therefore\ 2\cos\frac{3}{2}\beta\cos\frac{1}{2}\beta = 0$, $\therefore\ \beta = \frac{2}{3}(n\pi+\frac{1}{2}\pi)$, $2(n\pi+\frac{1}{2}\pi)$.

6. The figure is easy; in it $BD = c\dfrac{\sin\frac{1}{2}A}{\sin(\frac{1}{2}A+C)} = a\dfrac{\sin C}{\sin A}\dfrac{\sin\frac{1}{2}A}{\sin(\frac{1}{2}A+C)}$
$= \dfrac{a\sin C}{2\cos\frac{1}{2}A\sin(\frac{1}{2}A+C)} = \dfrac{a\sin C}{\sin(A+C)+\sin C} = \dfrac{a\sin C}{\sin B+\sin C}$.
Similarly for CD.

7. $\sin(\sin\theta) = \frac{1}{2} = \sin\frac{1}{6}\pi$, $\therefore\ \sin\theta = n\pi+(-1)^n\frac{1}{6}\pi$,
$\therefore\ \theta = \sin^{-1}[n\pi+(-1)^n\frac{1}{6}\pi]$.

8. When $\sin 2\theta$ is $+$ve, 2θ is between $2n\pi$ and $2n\pi+\pi$; $\therefore\ \theta$ is between $n\pi$ and $n\pi+\frac{1}{2}\pi$; and between these limits $\sin\theta$ and $\cos\theta$ are of like sign, and $\therefore$ their product $+$ve. When $\sin 2\theta$ is $-$ve, 2θ is between $2n\pi+\pi$ and $2n\pi+2\pi$; $\therefore\ \theta$ is between $n\pi+\frac{1}{2}\pi$ and $n\pi+\pi$; between these limits $\sin\theta$ and $\cos\theta$ are opposite in sign, and $\therefore$ their product $-$ve.

9. $\tan^{-1}x+\cot^{-1}x = n\pi+\tan^{-1}x+n\pi+\cot^{-1}x$
$= 2n\pi+\tan^{-1}x+\frac{1}{2}\pi-\tan^{-1}x = 2n\pi+\frac{1}{2}\pi.$

10. Let A, B (fig. 37) be the spectators, MH the vertical to the horizontal plane through A, B. Then $AB = 10$ m., $HBM = 45°$, $\therefore$ $BH = HM = x$ suppose; $HAM = 30°$, $\therefore\ AH = HM\cot 30° = x\sqrt{3}$. Also $BAH = 22\frac{1}{2}°$. Then $x^2 = 10^2+(\sqrt{3}x)^2-20\sqrt{3}x\cos 22\frac{1}{2}°$;
$\therefore\ 2x^2-10\sqrt{(6+3\sqrt{2})}\,x+100 = 0,$

$$\therefore\ x = \frac{10\sqrt{6+3\sqrt{2})} \pm \sqrt{[100(6+3\sqrt{2})-800]}}{4}$$

$= \frac{5}{2}[\sqrt{(6+3\sqrt{2})}\pm\sqrt{(3\sqrt{2}-2)}] = \frac{5}{2}[3{\cdot}2\pm1{\cdot}5]$ nearly $= 11\frac{3}{4}$ or $4\frac{1}{4}$.

11. Let p, P be perimeters of the figures. Then (Euc. IV. 15) radius of circle about hexagon $=\frac{1}{6}p$; and radius of circle in triangle

$$\left(=\frac{\Delta}{s}\right) = \frac{1}{2}\frac{P}{3}\cdot\frac{P}{3}\cdot\frac{\sqrt{3}}{2}\times\frac{2}{P} = \frac{P}{6\sqrt{3}},\qquad \therefore\ \pi\frac{36}{p^2} : \pi\frac{P^2}{36.3} = 3:4,$$

$\therefore\ p^2 : P^2 = 1:4,\quad \therefore\ p : P = 1:2.$

12. In fig. 38, suppose A, B, C the harbour, blockader, and ship when overtaken, respectively. Then $\angle B = 45°-23° = 22°$, $AB = 10$, $BC = 15$;

$$\therefore\ \tan\tfrac{1}{2}(A-C) = \frac{a-c}{a+c}\cot\tfrac{1}{2}B \qquad\qquad \begin{array}{rr} 3821 & 821 \\ \underline{1294} & \underline{777} \\ 2527 & 44 \end{array}$$

$$= \tfrac{5}{25}\cot 11° = \tfrac{2}{10}\cot 11°;$$

$$\therefore\ L\tan\tfrac{1}{2}(A-C) = \log 2+L\cot 11°-1 \qquad \tfrac{44}{2527}\text{ of }60'' = 1''.$$

$$= 10{\cdot}0123777;$$

$\therefore\ \frac{1}{2}(A-C) = 45°\,48'\,59''$, and $\frac{1}{2}(A+C) = 79°$;
adding, $A = 124°\,48'\,59''$, $\therefore$ course reqd. is $10°\,11'\,1''$ S. of E.

XXI.

1. $$\tan A+\cot A = \frac{\sin A}{\cos A}+\frac{\cos A}{\sin A} = \frac{\sin^2 A+\cos^2 A}{\sin A\cos A} = \frac{2}{\sin 2A};$$
and $\because\ \sin 2A$ is never >1, $\therefore\ \tan A+\cot A$ is never <2.

2. Let A be reqd. $\angle$. Then $L\tan A = L\cot A+{\cdot}5316768$
$= 10-\log\tan A+{\cdot}5316768 = 20-L\tan A+{\cdot}5316768,$
$\therefore\ L\tan A = 10{\cdot}2658384,\quad \therefore\ A = 61°\,32'.$

3. $(2bc\cos A)^2 = (b^2+c^2-a^2)^2 = b^4+c^4+a^4+2b^2c^2-2a^2b^2-2a^2c^2,$

$$\therefore\ 1-\cos^2 A = 1-\frac{b^4+c^4+a^2+2b^2c^2-2a^2b^2-2a^2c^2}{4b^2c^2}$$

$$= \frac{4b^2c^2-b^4-\&c.}{4b^2c^2},\quad \therefore\ f = l.$$

4. $2\cos^2\frac{1}{2}A = 1+\cos A = 1+\dfrac{b^2+c^2-a^2}{2bc} = \dfrac{(b+c)^2-a^2}{2bc}$,

and $2\sin^2\frac{1}{2}A = 1-\cos A = 1-\dfrac{b^2+c^2-a^2}{2bc} = \dfrac{a^2-(b-c)^2}{2bc}$,

$\therefore f = l$. When $a=b=c$, $A = 60^\circ$ and $\cot^2\frac{1}{2}A = 3$;

also $\dfrac{(b+c)^2-a^2}{a^2-(b-c)^2} = \dfrac{(2a)^2-a^2}{a^2-0} = \dfrac{3a^2}{a^2} = 3$.

5. Let a, b be any two chords in the same circle, respectively subtending at the circumference the angles α and β. Then, by Art. 138, if R be radius of the circle, $a = 2R\sin\alpha$, $b = 2R\sin\beta$, $\therefore a : b = \sin\alpha : \sin\beta$.

6. $\tan\theta = \sec\phi - \tan\phi = \dfrac{1-\sin\phi}{\cos\phi} = \dfrac{(\cos\frac{1}{2}\phi-\sin\frac{1}{2}\phi)^2}{\cos^2\frac{1}{2}\phi-\sin^2\frac{1}{2}\phi} = \dfrac{\cos\frac{1}{2}\phi-\sin\frac{1}{2}\phi}{\cos\frac{1}{2}\phi+\sin\frac{1}{2}\phi}$

$= \dfrac{1-\tan\frac{1}{2}\phi}{1+\tan\frac{1}{2}\phi} = \dfrac{\tan\frac{1}{4}\pi-\tan\frac{1}{2}\phi}{1+\tan\frac{1}{4}\pi\tan\frac{1}{2}\phi} = \tan(\frac{1}{4}\pi-\frac{1}{2}\phi)$,

$\therefore \theta = n\pi+\frac{1}{4}\pi-\frac{1}{2}\phi$.

7. $\sin 112^\circ$ is $> \cos 112^\circ$ and $+$ve; hence

$\sin 112^\circ + \cos 112^\circ = \sqrt{(1+\sin 224^\circ)} = \sqrt{(1-{\cdot}69)} = \sqrt{({\cdot}31)} = {\cdot}5568$,

$\sin 112^\circ - \cos 112^\circ = \sqrt{(1-\sin 224^\circ)} = \sqrt{(1+{\cdot}69)} = \sqrt{(1{\cdot}69)} = 1{\cdot}3$,

$\therefore 2\sin 112^\circ = 1{\cdot}8568$, and $\sin 112^\circ = {\cdot}9284$.

8. Let $\theta = \tan^{-1}\dfrac{x\sin\theta}{1+x\cos\theta}$, $\therefore \sec^2\theta = 1+\left(\dfrac{x\sin\theta}{1+x\cos\theta}\right)^2$

$= \dfrac{1+2x\cos\theta+x^2}{1+2x\cos\theta+x^2\cos^2\theta}$, $\therefore f = \cos 2\theta = 2\cos^2\theta - 1 =$

$\dfrac{2(1+2x\cos\theta+x^2\cos^2\theta)}{1+2x\cos\theta+x^2}-1 = \dfrac{1+2x\cos\theta+x^2(2\cos^2\theta-1)}{1+2x\cos\theta+x^2} = l$.

9. In fig. 39, let $AB = 40$ ft. $B = 90^\circ$, $A = 75^\circ$, $AO = OC = 25$. Then since B is in a semicircle, being right, $\therefore AC$ is a diameter; $\therefore ADC$ is right-angled; $\because AC = 50$ ft., $AB = 40$ ft., $\therefore BC = 30$ ft.,

$\therefore \triangle ABC = \frac{1}{2}\times 40\times 30$ sq. ft. $= 600$ sq. ft.; and if $CAB = \alpha$,

$\therefore \sin 2CAD = \sin(150^\circ-2\alpha) = \frac{1}{2}(2\cos^2\alpha-1)+\frac{1}{2}\sqrt{3}\,.\,2\sin\alpha\cos\alpha$

$= \frac{16}{25}-\frac{1}{2}+\sqrt{3}\,.\,\frac{3}{5}\,.\,\frac{4}{5} = \frac{1}{50}(7+24\sqrt{3})$,

$\therefore \triangle ADC = \frac{1}{2}AD\,.\,DC = \frac{1}{2}CA^2\sin CAD\,.\,\cos CAD = \frac{1}{4}(CA^2)\sin 2CAD$

$= \frac{1}{4}(50^2)\,.\,\frac{1}{50}(7+24\sqrt{3})$ sq. ft. $= \frac{50}{4}(7+24\sqrt{3})$ sq. ft.,

$\therefore$ area $ABCD = (600+87\frac{1}{2}+300\sqrt{3})$ sq. ft.

$= (687{\cdot}5+519{\cdot}6)$ sq. ft. $= 1207{\cdot}1$ sq. ft.

10. Let a be base. Now $s(s-a)(s-b)(s-c) = s(s-a)[s^2-(b+c)s+bc]$, where s, $s-a$ are constant; and $\therefore$ also $2s-a$ or $b+c$. Hence bc alone is variable. Now $4bc = (b+c)^2-(b-c)^2$, which is greatest when $b = c$; *i.e.*, when the triangle is isosceles.

11. In fig. 40, let D, E be respective points of contact of inscribed and escribed circles with BC. Then $CD \sim CE = OD\cot\frac{1}{2}C \sim O'E\tan\frac{1}{2}C$

$= \dfrac{\Delta}{s}\sqrt{\dfrac{s(s-c)}{(s-a)(s-b)}} \sim \dfrac{\Delta}{s-a}\sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}} = (s-c)\sim(s-b) = b\sim c$.

12. (1) $\because$ the elevations are all equal, the points of observation are equally distant from the foot of the tower. This latter therefore is the centre of the circle passing through those points. Hence distance = radius of circle about the triangle formed by the given distances, and may be known by formula $r = \frac{\Delta}{s}$. The required height is perpendicular in a right-angled triangle whose base and base-angle are known.

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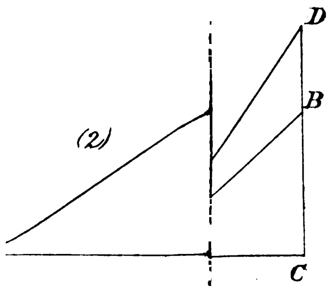
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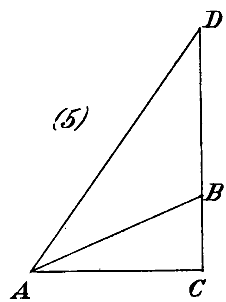
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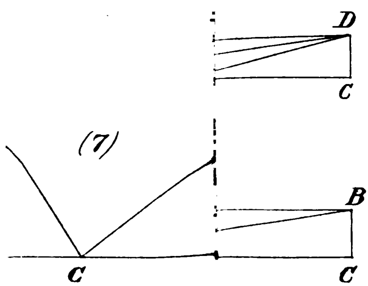
D
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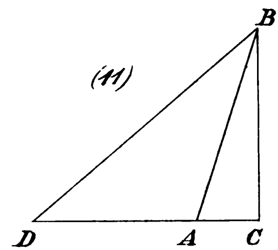
(7)
B
C
C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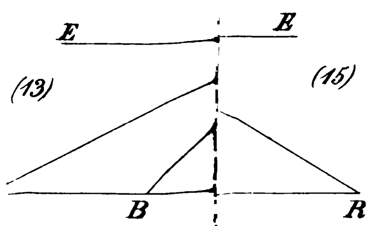
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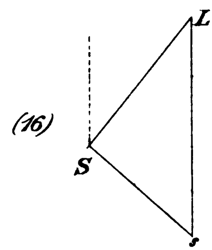
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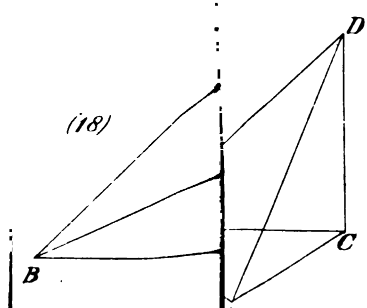
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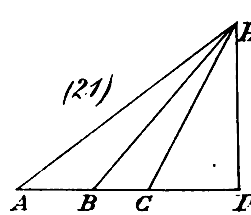
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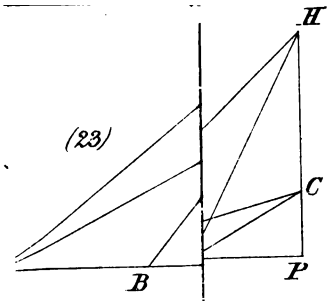
A
B
C
D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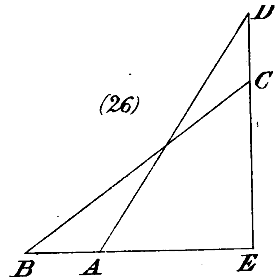
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D
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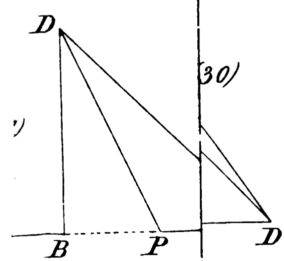
E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B
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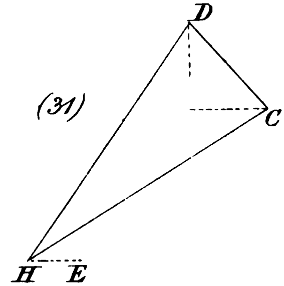
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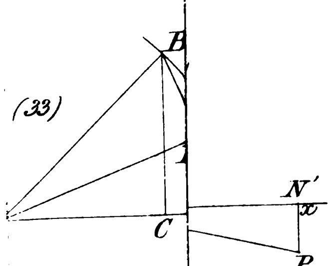
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N′
x
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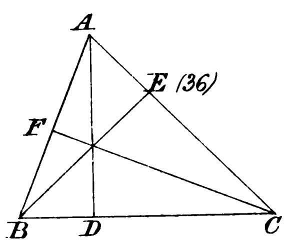
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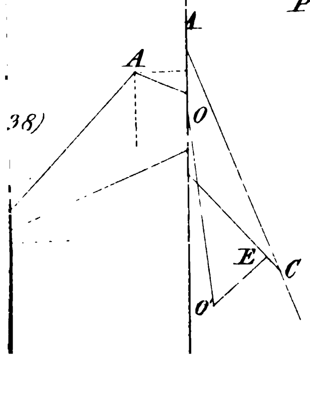
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